THE SPRING MEETING OF THE CHICAGO SECTION.

THE thirty-first regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, March 21–22, 1913, extending through four half-day sessions. The total attendance was eighty-one, including the following fifty-one members of the Society:

Professor R. P. Baker, Professor G. N. Bauer, Professor G. A. Bliss, Dr. R. L. Börger, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor E. W. Davis, Dr. W. W. Denton, Professor L. E. Dickson, Mr. C. R. Dines, Professor Arnold Dresden, Professor O. E. Glenn, Dr. T. H. Gronwall, Dr. T. H. Hildebrandt, Professor G. W. Hartwell, Mr. W. C. Krathwohl, Professor Kurt Laves, Professor A. C. Lunn, Dr. E. B. Lytle, Professor H. W. March, Professor W. D. MacMillan, Professor E. H. Moore, Professor F. R. Moulton, Mr. E. J. Moulton, Dr. Anna J. Pell, Professor Alexander Pell, Professor H. L. Rietz, Professor W. H. Roever, Dr. R. E. Root, Mr. A. R. Schweitzer, Miss Ida M. Schottenfels, Professor J. B. Shaw, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Dr. E. B. Stouffer, Principal F. C. Touton, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor C. A. Waldo, Miss Mary E. Wells, Mr. C. W. Wester, Professor E. J. Wilczynski, Professor D. T. Wilson, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor E. B. Van Vleck, President of the Society, presided at the two morning sessions and Professor D. R. Curtiss, Chairman of the Section, at the two afternoon sessions.

On Friday evening about fifty members dined together at the Quadrangle Club of the University, after which some attended a club lecture by Professor R. D. Salisbury, of the department of geography of the University of Chicago, while others spent the time in social intercourse in the club rooms.

At the business meeting on Friday afternoon the following

officers of the Section were elected to hold office till the December meeting, 1913: Professor D. R. Curtiss, Chairman; Professor H. E. Slaught, Secretary; Professor A. L. Underhill, member of the program committee. The election of officers regularly occurs at the December meeting, but was not held in 1912 on account of the merging of the Section meeting in that of the Society at Cleveland. In connection with the election, the following resolution was unanimously adopted: That the Section favors the continuation of the custom of having a formal address from the Chairman upon his retirement from office. It was felt that there are too few papers giving a general survey of any field in mathematics and that this custom would insure at least a periodic opportunity of this kind.

Under the head of "informal notes and queries" several questions were briefly discussed, including a statement by Professor W. H. Roever with respect to the drawing of figures in space and the need by mathematicians of a better understanding of the principles involved. This need was acknowledged and the hope was expressed that we might have some adequate texts in English which could be used in outlining a course in this subject.

The following papers were presented at this meeting:

(1) Dr. T. H. Gronwall: "On various summation methods and their application to Fourier's series."

(2) Dr. R. E. Root: "Limits in terms of order, with example of limiting element not approachable by a sequence."

(3) Professor L. C. Karpinski: "John Caswell."

(4) Professor Karpinski: "The Whetstone of Witte."

(5) Professor R. P. Baker: "The topology of logical diagrams."

(6) Professor Baker: "The construction of the lines of a complex from given lines by ruler only."

(7) Professor W. O. Beal: "Concerning the stability of the eighth satellite of Jupiter."

(8) Professor G. A. Miller: "On the representation groups of given abstract groups."

(9) Miss Mildred L. Sanderson: "A fundamental theorem in the theory of modular invariants" (preliminary report).

(10) Mr. W. L. Miser: "On the solutions of linear homogeneous differential equations with elliptic function coefficients" (preliminary report).

(11) PROFESSOR R. D. CARMICHAEL: "On the Brownian movement and the ultimate constitution of matter."

(12) PROFESSOR CARMICHAEL: "On the impossibility of certain diophantine equations and systems of equations."

(13) PROFESSOR CARMICHAEL: "On certain diophantine equations having double parameter solutions."

(14) PROFESSOR F. R. MOULTON: "On orbits of ejection and collision in the problem of three bodies."

(15) PROFESSOR J. B. SHAW: "Formal determination of Clifford algebras."

(16) PROFESSOR E. J. WILCZYNSKI: "A new type of integral equations" (preliminary report).

(17) DR. E. W. CHITTENDEN and PROFESSOR A. D. PITCHER: "Classes which admit a development."

(18) DR. E. W. CHITTENDEN: "Relatively uniform convergence of series of functions."

(19) PROFESSOR G. N. BAUER and DR. H. L. SLOBIN: "Some transcendental curves and numbers."

(20) MR. C. E. LOVE: "Irregular integrals of the linear differential equation of the third order."

(21) MISS IDA M. SCHOTTENFELS: "A set of generators for quaternary linear groups."

(22) PROFESSOR G. A. BLISS: "A method of subdividing the area enclosed by a plane curve, with an application to Cauchy's theorem."

(23) DR. S. LEFSCHETZ: "The base for algebraic $(r-1)$-spreads immersed in an r-spread, with some applications."

(24) PROFESSOR E. L. DODD: "A justification of empirical probability based upon an unknown a priori probability."

(25) MR. K. P. WILLIAMS: "Concerning second order difference equations and the Schwarzian difference."

(26) PROFESSOR A. C. LUNN: "Integral equations in the kinetic theory of gases."

(27) PROFESSOR LUNN: "An integral functional equation in the theory of Brownian movements."

(28) PROFESSOR L. E. DICKSON: "Proof of the finiteness of modular covariants."

(29) PROFESSOR DICKSON: "On the rank of a symmetrical matrix."

(30) PROFESSOR DICKSON: "The projective geometry and covariant theory of a ternary quadratic form modulo 2."

(31) MR. J. MCDONALD: "On quadratic residues."

(32) PROFESSOR E. H. MOORE: "On the geometry of linear homogeneous transformations of m variables" (preliminary communication).

(33) PROFESSOR O. D. KELLOGG: "Conditions for a certain nomographic representation of a function of three variables."

(34) PROFESSOR O. E. GLENN: "Symbolic theory of finite expansions."

(35) PROFESSOR KURT LAVES: "Concerning a complete theory of the motion of the four minor satellites of Saturn."

(36) PROFESSOR MAXIME BÔCHER: "An application of Gibbs's phenomenon."

(37) MR. A. R. SCHWEITZER: "On the working hypothesis in the genetic logic of mathematics. Second paper."

(38) MR. A. R. SCHWEITZER: "A seeming contradiction in Poincaré's logical position."

(39) DR. A. R. CRATHORNE: "The total variation of the isoperimetric problem with variable end points."

The papers of Miss Sanderson and Mr. MacDonald were communicated to the Society through Professor Dickson, and those of Professor Beal and Mr. Miser through Professor Moulton. In the absence of the authors the papers of Professors Karpinski, Miller, Carmichael, Dodd, Bôcher, Kellogg, Mr. Love, Dr. Lefschetz, Mr. Williams, Mr. MacDonald, and Dr. Crathorne were read by title. Abstracts of the papers follow in the order of the titles given above.

1. De la Vallée Poussin has defined the sum of any series $u_0 + u_1 + \cdots + u_n + \cdots$ as the limit (when it exists) for $n = \infty$ of the expression

$$V_n = u_0 + \sum_{v=1}^{n} \frac{n(n-1)\cdots(n-v+1)}{(n+1)(n+2)\cdots(n+v)} u_v,$$

and shows that when a function $f(x)$ has a generalized derivative of order k at a point x_0, then at this point the kth derivatives of the partial sums V_n of the Fourier series corresponding to $f(x)$ tend toward the kth generalized derivative of $f(x)$ for $n = \infty$ (*Bulletin de l'Académie Belgique des Sciences*, 1908).

In the present paper, Dr. Gronwall studies the relation between this summation method and that of Cesàro founded upon successive arithmetical means; the following propositions are established:

I. When a series $u_0 + u_1 + \cdots + u_n + \cdots$ is summable by Cesàro's means of any given order r, it is also summable by de la Vallée Poussin's process, with the same sum.

II. There exist series summable by de la Vallée Poussin's process, which are not summable by Cesàro's means of any finite order.

III. When $f(x)$ has a generalized derivative of order k at $x = x_0$, then at this point the kth derivatives of the Cesàro means of order $k + 1$ of the Fourier series corresponding to $f(x)$ converge toward the kth generalized derivative of $f(x)$.

Thus de la Vallée Poussin's theorem is implied by III (on account of I), but does not imply III (by reason of II).

2. While the conditions used in the paper by Dr. Root are fulfilled by any simply ordered class, provision is made for multiple interpretation of its postulates and theorems. The work pertains to a class of elements q of arbitrary character with a relation B of the type of "betweenness." The properties postulated for this relation are such as to persist under composition of systems. The relation B replaces betweenness in the definition of segment, and an element q is said to be a limiting element of any set that has an element distinct from q in every segment containing q. Certain fundamental theorems, including the proposition that every derived class is closed, result from the type of the system, without postulates. Very mild assumptions lead to a theory of multiple and iterated limits of functions, including as a special case the usual theorems on sequences of continuous functions. An additional postulate leads to a form of the Heine-Borel theorem, and to theorems on bounds of functions on a compact class. A sequence of elements is said to have the limit q if every segment containing q contains all but a finite number of elements of the sequence. On the basis of the postulates this definition of limit fulfils the conditions specified by Fréchet, thus securing, by means of the sequential definition of limiting element of class, a considerable body of theorems developed by Fréchet, Hedrick, and Hildebrandt. There arise, then, two parallel theories, of about equal extent, one based on the neighborhood definition of limiting element, the other on the sequential definition. That these two theories are not equivalent is shown by an example of a simply ordered class fulfilling all the postulates of the paper

and possessing a limiting element that is not the limit of any sequence of elements of the class.

3. Little has been known about the life of the English mathematician John Caswell (1655–1712), a contemporary of Wallis and the author of a somewhat notable work on trigonometry. Professor Karpinski has gathered the facts of his life largely from the publications of the Oxford Historical Society. Some light is thrown also on the general conditions with respect to the study of mathematics in the English universities during the seventeenth and early eighteenth centuries. The paper is to appear in the *Bibliotheca Mathematica.*

4. Some time ago the editor of the *Bibliotheca Mathematica* suggested the desirability of a study of Robert Recorde's "The Whetstone of Witte," with a view to determining the real contributions of this work to the development of algebra and algebraic symbolism. Professor Karpinski has made such a study. The net result would seem to be that the most noteworthy contribution of Recorde was the introduction of the equality sign. The works of Stifel, Scheybl, and Cardan were largely drawn upon by him. However Recorde contributed to the advance of the study of mathematics by furnishing in the English language excellent treatises from a pedagogical point of view.

5. In this paper Professor Baker shows that in a finite logical field two features are desirable, namely, integrity of the class regions and integrity of neighborhood. With line segments, as used by some writers, the first property is not available for $n > 2$. The second fails when areas are used when $n \geqq 4$. The first can always be obtained with areas. The plane representation is of lower genus than the Dyck representation of the corresponding abstract group. This reduction of genus occurs in other groups, which cannot be generated by two elements. The redundant generators used by Dyck in the case of infinite groups are not needed in finite groups, and they raise the genus. The minimum genus is given by a Cayley color diagram.

6. Plücker's construction of a complex, followed by many texts, uses the hyperboloid determined by three lines and its

intersections with one of the others. Cases exist in which all possible intersections are complex. To avoid this is desirable. The ruler construction which Professor Baker uses is based on incidences of lines and uses only planes of constructional lines.

7. Using the elements of the orbit of the eighth satellite of Jupiter given by Crommelin in volume 75, page 50 et seq., of the *Monthly Notices*, Professor Beal shows by applying the well-known criterion of stability which follows from Jacobi's integral, that the motion is stable, that is, that this satellite cannot become an independent asteroid. This is contrary to the conclusion reached by Kobb in the *Bulletin Astronomique* of 1908, page 414, from the first provisional elements that were given out from the Greenwich Observatory.

8. The abstract group G is said to be a representation group (Darstellungsgruppe) of the finite abstract group G', whenever G is one of the largest groups involving a quotient group which is simply isomorphic with G' as regards a subgroup M that is both in the central and also in the commutator subgroup of G. A number of fundamental theorems relating to representation groups were developed by I. Schur in two memoirs published in volumes 127 and 132 of *Crelle*. The present paper aims to establish some new results relating to these groups, and to derive some of the known theorems by simpler methods. Additional relations existing between the theory of representation groups and known theorems of abstract groups are also developed, especially as regards metabelian groups.

Among the theorems which have been established by Professor Miller are the following: Every possible group contains at least one commutator besides identity which is commutative with at least one of its elements. Every non-cyclic abelian group has at least two distinct representation groups, and there are infinite systems of representation groups of abelian groups whose commutator subgroups involve operators which are not commutators. If the square of an operator is commutative with another operator, their commutator is transformed into its inverse by the former of these two operators, and if a commutator is commutative with one of its two elements, its order is a divisor of the index of the lowest

power to which this element must be raised to be commutative with the other element. The order of every representation group of a group of order p^m, p being a prime number, is of the form $p^{m'}$, and the orders of all of its commutators are divisors of p^{m-1}.

9. Miss Sanderson proves the following theorem: To any modular invariant i, of any system of forms under any group G of linear transformations with coefficients in the $GF(p^n)$, corresponds a function I of arbitrary variables formally invariant under G, and such that $I = i$ for all sets of values of the variables in the $GF(p^n)$. The proof rests upon the lemma: Let $a_1, \cdots, a_r$ be r arbitrary variables ($r > 1$), and $g_1, \cdots, g_r$ given elements of the $GF(p^n)$, $g_r \neq 0$. Then the determinants

$$N = \begin{vmatrix} a_1, & \cdots, & a_r \\ a_1^{p^n}, & \cdots, & a_r^{p^n} \\ \cdots & \cdots & \cdots \\ a_1^{p^{(r-1)n}}, & \cdots, & a_r^{p^{(r-1)n}} \end{vmatrix}, \quad D = \begin{vmatrix} a_1, & \cdots, & a^r \\ \cdots & \cdots & \cdots \\ a_1^{p^{(r-2)n}}, & \cdots, & a_r^{p^{(r-2)n}} \\ g_1, & \cdots, & g_r \end{vmatrix}$$

are such that N is divisible by D in the field and the quotient $Q = N/D$ has the properties:

$Q \neq 0$ if $a_1 = g_1, \cdots, a_r = g_r$; $Q = 0$ if $a_1 = e_1, \cdots, a_r = e_r$, where $e_1, \cdots, e_r$ are elements of the field not proportional to $g_1, \cdots, g_r$.

If $a_1, \cdots, a_r$ are the coefficients of the forms, a fundamental system of invariants which are formally invariant is formed out of sums as to $g_1, \cdots, g_r$ of powers of these Q's, where the $g_1, \cdots, g_r$ are the coefficients of a particular set of forms in the system.

10. In this paper Mr. Miser treats of the forms and properties of the multiple-valued solutions of the linear homogeneous differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \psi_{ij}(t) x_j \qquad (i = 1, \cdots, n),$$

where the $\psi_{ij}(t)$ are elliptic functions whose primitive periods are ω, ω', and whose poles are of order one. When the n roots of the indicial equation for each singular point are all distinct modulo unity, it is shown how single-valued doubly

periodic functions $\phi_j(t)$ can be determined so that the general solution is

$$x_i = \sum_{j=1}^{n} A_j e^{\int \phi_j(t)dt} \xi_{ij}(t) \qquad (i = 1, \cdots, n),$$

where the ξ_{ij} are regular functions in the whole finite part of the plane.

11. In this paper Professor Carmichael proposes a new law of action among the ultimate mechanical units of matter. The law is suggested by the behavior of the Brownian particles as they are observed under the microscope, and especially under the ultra-microscope. As an analytical expression of the law we have the following: As the distance x between any two particles of matter varies, the force of attraction varies as $f(x)$, where

$$f(x) = \frac{1}{x^2} + \frac{a_4}{x^4} + \frac{a_6}{x^6} + \frac{a_8}{x^8} + \cdots,$$

$a_4, a_6, a_8, \cdots$ being properly determined constants. These constants are in general different for different kinds of matter. Corresponding to the fact that a particular kind of matter may in general exist in three and but three forms, namely, as a solid, a liquid, a gas, are certain relations among the constants $a_4, a_6, \cdots$. Other fundamental properties of matter also have their analytic correspondents; as, for instance, the fact that all gases have so nearly the same physical properties. The value of this law in the explanation of the processes of nature will depend on how well it lends itself to the use of the experimenter. Immediately it suggests numerous tests to be applied in the laboratory and it would thus appear to be an instrument of value.

12. The object of Professor Carmichael's second paper is to prove the impossibility of certain interesting diophantine equations and systems of equations. In two cases the argument is carried out by means of Fermat's famous method of "infinite descent." The paper will be offered for publication in the *American Mathematical Monthly*.

13. In his third paper Professor Carmichael obtains a double

parameter solution of each of several diophantine equations of which the following are typical:

$$x^3 + 2y^3 + 3z^3 = t^3, \quad x^3 + y^3 + z^3 = 2t^3,$$
$$x^3 + y^3 + z^3 + u^3 = 3t^3, \quad x^4 + y^4 + 4z^4 = t^4.$$

14. The problem under consideration by Professor Moulton is that in which an infinitesimal body moves subject to the attraction of two finite masses. Orbits of ejection and collision are those in which the infinitesimal body leaves one of the finite bodies and collides with the other one. The existence of infinitely many of these orbits is established. They are of importance in the consideration of the escape of molecules from the atmosphere of one body to another, and also in the consideration of periodic orbits, because certain types of these orbits are the limits of families of periodic orbits.

15. Professor Shaw's paper is a deduction of the properties of a Clifford algebra of N dimensions from the defining equation $\alpha\beta = -\beta\alpha + I\alpha\beta$. A Clifford algebra is one whose units consist of a fundamental set $i_1, \cdots, i_n$, their products two at a time, three at a time, etc., with the equations

$$i_s^2 = -1 \quad (s = 1, \cdots, n), \qquad i_s i_t = -i_t i_s \quad (s \neq t).$$

16. The integral equations considered by Professor Wilczynski are linear equations, both of the Volterra and the Fredholm form, and include equations of the first as well as of the second kind. The novelty consists in the form of the integral which is regarded, in the simplest case, as an open line integral independent of the path.

17. In terms of a general class P which admits a development Δ as defined by E. H. Moore, Drs. Pitcher and Chittenden have developed a theory analogous to that of Fréchet, in which the results obtained by Fréchet, Hedrick, Hildebrandt and others are secured through appropriate conditions on the development Δ of P.

18. Dr. Chittenden considers the immediate consequences of the definition of relatively uniform convergence of series of functions given by E. H. Moore, and shows that if the set

Q of points, at which the measure of convergence of a series of functions defined on an interval (a, b) is greater than zero, is reducible, the series converges relatively uniformly as to a scale which admits a definition in terms of the given series and set Q. He shows further that if for a series of continuous functions there is a perfect set on which the set Q is dense, then the series converges relatively uniformly as to no scale function. Osgood has given examples of such series.

19. In this paper Professor Bauer and Dr. Slobin trace some of the consequences of Lindemann's theorem that in the equation $e^x = y$, x and y cannot both be algebraic numbers (*Mathematische Annalen*, volume 20, page 224). Let $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y)$ be of the form $P_0y^n + P_1y^{n-1} + \cdots + P_n$, where the P's are polynomials in x, and where all the constants involved are algebraic numbers; further let there be no common factors in φ_1, φ_2, and φ_3. Then the following theorems are established: In the equation $\varphi_1(x, y) \cdot e^{\varphi_2(x, y)} + \varphi_3(x, y) = 0$, x and y cannot both be algebraic numbers, except for a finite number of pairs of values determined by the simultaneous equations

$$\varphi_1(x, y) = 0, \qquad \varphi_2(x, y) = 0$$

and

$$\varphi_1(x, y) = 0, \qquad \varphi_1(x, y) + \varphi_3(x, y) = 0.$$

All the direct and inverse trigonometric and hyperbolic functions represent transcendental numbers for all algebraic values of the argument, except zero. Equations of the form

$$\varphi_1(x, y)e^{\varphi_2(x, y)} + \varphi_3(x, y) = 0,$$

$$\varphi_1(x, y) + \varphi_2(x, y) \tanh \varphi_3(x, y) = 0,$$

$$\varphi_1(x, y) + t = 0,$$

where t is any transcendental number and where, instead of the hyperbolic tangent, any other direct or inverse circular or hyperbolic function may be substituted, represent families of curves which pass at most through a finite number of algebraic points.

By expanding $f_1(x)$, $e^{f_2(x)}$, $f_1(x) \log f_2(x)$, $f_1(x) \sin f_2(x)$, etc., where $f_1(x)$, $f_2(x)$ are explicit algebraic functions of x, series are obtained which represent transcendental numbers whenever an algebraic number is substituted for x.

20. Mr. Love's paper considers the existence and form of asymptotic solutions for the differential equation

$$y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0,$$

where the coefficients are developable, for large real values of x, in asymptotic (or convergent) series of the form

$$a_r(x) \sim x^{rk}\left(a_{r,0} + \frac{a_{r,1}}{x} + \cdots\right) \quad (r = 1, 2, 3).$$

Detailed study is made of the various cases in which the so-called characteristic equation has multiple roots, since these cases do not appear to have been treated heretofore. The corresponding problem for the equation of second order has been considered in an earlier paper of which the present work forms a continuation.

21. In volume 12 (1905) of the BULLETIN Miss Schottenfels published the paper, "A set of generators for ternary linear groups." In the present paper she develops a similar set of generators for quaternary linear groups of the type $x_i = x_{i+1}$, $x_k = x_1 + hx_2$ $(i = 1, 2, \cdots, k-1;\ h = 0, 1)$.* It is proved that all substitutions of the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4, \\ x_4' &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4, \end{aligned}$$

with coefficients rational integers of determinant unity, can be formed by combinations of the substitutions

$$T = (x_2 \cdot x_3 \cdot x_4 \cdot x_1), \quad S = (x_2 \cdot x_3 \cdot x_4 \cdot x_2 + x_1).$$

It is also proved that all matrices of the form

$$(\alpha_{ij}) = \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{matrix} \quad ,j = 1, 2, 3, 4,)$$

* Moore, *Mathematische Annalen*, vol. 51, p. 436.
Schottenfels, *Annals of Mathematics*, ser. 2, vol. 1, no. 8. BULLETIN, VOL 6, pp. 440–443.
Burnside, *Messenger of Mathematics*, vol. 24, p. 109.

where α_{ij} are marks of the GF $[2^1]$ and (α_{ij}) is of determinant unity, can be formed by combinations of

$$T = \begin{matrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1' \\ 1\,0\,0\,0 \end{matrix} \quad S = \begin{matrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1' \\ 1\,1\,0\,0 \end{matrix}$$

where T and S generate a group holoedrically isomorphic to the alternating group upon eight letters, $G_{8!/2}$.

22. Consider a simply closed continuous rectifiable curve which divides the plane into an exterior and an interior. Professor Bliss shows that the interior of such a curve can be divided by a finite number of segments of straight lines into regions whose diameters are less than ϵ, the number of auxiliary segments necessary being not greater than $4^{L\sqrt{2}/\epsilon}$. The theorem has a number of applications, one of which may be of special interest. With its help Cauchy's theorem in the theory of functions can be easily proved for any rectifiable curve if it has first been deduced for a rectangle. The methods of Goursat and Moore are readily applicable to a rectangle, but involve difficulties when applied to a curve which is supposed only to be rectifiable, on account of the complicated way in which a rectifiable curve may intersect a straight line.

23. In this paper Dr. Lefschetz gives the extension to an r-spread V_r of the theory of the minimum base developed by Severi and Poincaré for algebraic surfaces. Let $\{\Phi\}$ and $\{\Psi\}$ be two continuous systems, ∞^2 at least, of $(r-1)$-spreads immersed in V_r, $\{\varphi\}$ and $\{\psi\}$ the systems, ∞^2 at least, which they cut out on the general variety of a system $\{\Pi\}$. Then if φ and ψ are equivalent on Π when the latter varies, the same holds for Φ and Ψ, that is, there exists a continuous system in V_r containing them both totally. If ρ, ρ' and σ, σ' are the numbers of Picard and Severi for V_r and Π respectively, then $\rho \leqq \rho'$, $\sigma \leqq \sigma'$, from which follows the existence of a minimum base on V_r. Hence if, for any system such as $\{\Pi\}$, $\rho' = \sigma' = 1$, we have also $\rho = \sigma = 1$.

In particular if V_r has no singularities whatever, then all the $(r-1)$-spreads immersed in it are complete intersections — a known result. Also if V_r is the locus of ∞^1 $(r-1)$-flats then $\rho = 2$, $\sigma = 1$. Hence if ϕ_i is an $(r-1)$-spread in such a

V_r, a_i its order, α_i the order of its section by the flats, N the number of points common to r such spreads in V_r, then Story's generalized formula,

$$N = \Sigma a_1\alpha_2\cdots\alpha_r - (r-1)\mu\prod_1^r\alpha_i,$$

follows at once. Finally a new invariant number is introduced by considering the base cut out in the canonic system of V_r, and it is shown to exist even if V_r has no canonic system.

24. If an insurance company finds that, of 1,000 men of a given age, 990 are alive at the end of one year, what justifies the acceptance of .99 as the probability that a man of the given age will live at least a year? Dr. Dodd uses an unknown a priori probability in the following theorems. (1) Suppose that an event of unknown probability p has occurred just r times in s trials. Let the probability a priori that p lies in the interval $I = (x_1, x_1 + \delta)$ be $\int_{x_1}^{x_1+\delta}\psi(x)dx$; where $\psi(x)$ is limited and integrable; and, moreover, has a positive minimum in (0, 1), or at least in that portion of $(\sigma, 1-\sigma)$ which remains when a finite number of sub-intervals such as $(\xi-\sigma, \xi+\sigma)$ are removed, σ being taken small at pleasure. Then the probability a posteriori that p lies in I is greatest, with a chosen δ, when $x_1 = (r/s) + \eta$, where $\lim_{s=\infty,\,\delta=0}\eta = 0$, uniformly with respect to r. Roughly speaking: The most probable value of p is r/s when s is large enough. (2) If $\psi(x)$ satisfies the conditions specified in (1), the probable value of p will differ from r/s by less than any preassigned positive ϵ when s is large enough. (3) If $\psi(x)$ satisfies the conditions of (1), and s_1 additional (future) trials are to be made, then the most probable number of future occurrences is one (or possibly both—so to speak) of the two integers nearest to $(r/s)s_1$, provided that s, the number of past trials, is large enough.

25. Mr. Williams determines a difference equation of the third order which is satisfied by the quotient of any two solutions of a homogeneous linear second order difference equation. This third order equation has the property that, if any particular solution of it can be obtained, then the problem of solving the second order equation is reduced to finding a solution of a first order equation. An expression analogous

to the Schwarzian derivative of a function is obtained, and it is shown in a simple direct manner that this expression is invariant for a linear fractional transformation with periodic coefficients.

26. A fundamental problem of the kinetic theory of gases is the determination of the distribution of the molecules in space and velocity, by solution of Boltzmann's differential-integral equation for assigned physical conditions. For the case where the molecules are thought of as hard elastic spheres Hilbert has recently shown that, if the solution be sought in the form of a power series in a guiding parameter, each step of the computation corresponds to the solution of an integral equation of the Fredholm type with symmetric kernel.

The first part of Professor Lunn's paper gives another proof of this reduction, of a more geometric character, extending the theorem, moreover, so as to assume simply that the molecules have kinetic spherical symmetry, that the mutual forces during an encounter are such as to conserve momentum and energy, and never lead to union of two molecules, and that a certain condition of convergence is satisfied. In all cases the kernel is a function of the absolute values of two local molecular velocities and of the angle δ between them.

The second part of the paper shows that if the kernel be thought of as expanded in a series of Legendre polynomials in $\cos\delta$, and the known functions in the integral equations in series of surface harmonics in the velocity space, then the corresponding expansions of the unknown functions are determined by the solution of certain Fredholm equations in the one-dimensional realm of the absolute value of the velocity.

The first term of the power series was shown by Hilbert to be the well-known Maxwell formula. The general form of the second term is here found to contain, besides the additive solutions of the homogeneous equations, only harmonics of the first and second orders, corresponding respectively to thermal gradient and viscous stress, and leading to a computation of the coefficients of thermal conduction and viscosity. The value of the latter coefficient given by Meyer, and that of the former resulting from one of Tait's equations, which seems to be little known but fits experimental conditions better than Meyer's, both prove to be only the first terms in series resulting from solution of the integral equations by iteration.

27. In this paper Professor Lunn points out an obscurity in Einstein's treatment of the theory of Brownian movements, which consists in reducing the problem to the differential equation of diffusion. An alternative treatment is suggested, based on a kind of property of transitivity of the distribution function, leading to a non-linear integral functional equation, of which the Fourier-Einstein formula gives a special solution. Other special solutions are found, such as to show that the integral equation alone does not imply either the differential equation or certain auxiliary conditions suggested by the physical problem. The question is raised whether the solution is determinate when these other conditions are included.

28. In the first paper by Professor Dickson, it is shown that all rational integral modular covariants of a system of forms in n variables are rational integral functions of a finite number of such modular covariants. The proof makes use of the universal covariants* of the general linear modular group and of a lemma† which states that any set of monomial functions of n variables contains a finite number of functions $M_1, \cdots, M_f$, such that any function of the set is the product of some M_i by a monomial function. The paper will appear in the *Transactions*.

29. The second paper by Professor Dickson gives a short, elementary proof of Kronecker's theorem that in a symmetrical matrix of rank $r > 0$ at least one principal minor of order r is not zero.

30. In the third paper by Professor Dickson, certain invariants and covariants of the ternary form $F = b_1x_1^2 + \cdots + a_1x_2x_3 + \cdots$ modulo 2 (later shown to form a fundamental system) were obtained by a study of the (infinite) projective geometry in which the homogeneous coordinates x_1, x_2, x_3 of a point are roots of any congruences modulo 2 with integral coefficients. The polar of the point (y) with respect to F is given by a determinant whose rows are a_1, a_2, a_3; y_1, y_2, y_3; and z_1, z_2, z_3. The polar thus passes through (y) and the apex (a) of the conic $F \equiv 0$. Any line through (a) is a tangent to the conic, whose line equation is thus $\kappa = a_1u_1 + a_2u_2$

* *Transactions*, vol. 12 (1911), p. 75.
† *Amer. Journal of Math.*, October, 1913.

$+ a_3u_3 = 0$. Since the a's are cogredient with the x's, the function Δ obtained by replacing x_i by a_i in F is an invariant, the discriminant of F. In the polar form, take $(y) = (x)$, $z = (x^2)$. Then

$$K = \begin{vmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$$

is a covariant of F. For the typical non-degenerate form $F = x_1x_2 + x_3^2$, we have $K = x_1x_2(x_1 + x_2)$, and the resulting three lines meet the conic in three points, which with the apex (0, 0, 1) define a complete quadrangle covariantly associated with the conic. The diagonal points of this quadrangle lie on the line $x_1 + x_2 + x_3 \equiv 0 \pmod 2$. Applying a linear transformation which replaces the special form F by the general one having $\Delta = 1$, the special line yields the linear covariant

$$L = \Sigma(\beta_i + 1)x_i, \qquad \beta_1 = b_1 + a_2a_3 + a_2 + a_3, \ \cdots.$$

The invariant $\beta_1\beta_2\beta_3$ of L is an invariant of F. Forming the corresponding function for $F + \lambda\kappa^2$, we obtain an invariant of F and κ, from which we deduce a formal contravariant of the second order of F. L has the same relation to the latter that κ had to F.

31. The object of Mr. McDonald's paper is: (1) To give a direct proof of the reciprocity theorem in the case of the number 2 based on the binomial theorem; (2) To show how the quadratic character of an odd number m with respect to an odd prime n may be determined, from the binomial theorem, without the separation of m into its factors; (3) To exhibit the identity between a certain function of m and n and the Legendre-Jacobi symbol (m/n), when m and n are odd numbers.

32. Let a linear homogeneous transformation T of the m-space of m cartesian coordinates have the s distinct roots w_g of respective multiplicity m_g $(g = 1, \cdots, s)$; the determinant of the transformation $T - wI$ (I = identity) is $(w_1 - w)^{m_1} \cdots (w_s - w)^{m_s}$ with $m = m_1 + \cdots + m_s$. The following canonical form of the transformation T is well known:

$$T\varphi_{ghij} = \varphi_{ghij-1} + w_g\varphi_{ghij},$$

where the φ_{ghij} are certain m linearly independent points or vectors of the m-space: $g = 1, \cdots, s$; $h = 1, \cdots, k_g$; $i = 1, \cdots, l_{gh}$; $j = 1, \cdots, n_{gh}$, with $n_{g1} > \cdots > n_{gk_g}$ and $l_{g1}n_{g1} + \cdots + l_{gk_g}n_{gk_g} = m_g$; further $\varphi_{ghi0} = 0$. The vectors φ_{ghij} form a basis of the m-space canonical for the transformation T. The positive integers k_g, l_{gh}, n_{gh} are the same for all canonical bases (φ_{ghij}).

In his note Professor Moore determines the linear $\overline{m}$-spaces of the m-space which are invariant as to choice of canonical basis, that is, spaces such that, if for a canonical basis (φ_{ghij}) the vector $\Sigma_{ghij}a_{ghij}\varphi_{ghij}$ belongs to the $\overline{m}$-space, so does the vector $\Sigma_{ghij}a_{ghij}\varphi'_{ghij}$ for every canonical basis (φ'_{ghij}). Such an $\overline{m}$-space is invariant under T, or, if T is singular, it at least transforms by T into a sub-space of itself. For the $\overline{m}$-space the roots w_g are of multiplicity $\overline{m}_g$ $(0 \leqq \overline{m}_g \leqq m_g)$ and a canonical basis $(\overline{\varphi}_{ghij})$ is a part or all of a canonical basis of the m-space, viz., $\overline{\varphi}_{ghij} = \varphi_{ghij}$ for $g = 1, \cdots, s$ with $\overline{m}_g \neq 0$, $h = 1, \cdots, k_g$; $i = 1, \cdots, l_{gh}$; $j = 1, \cdots, \overline{n}_{gh}$, where the numbers $\overline{n}_{gh}$ are subject to the conditions

$$0 \leqq \overline{n}_{gh} \leqq n_{gh};$$

$$\overline{n}_{g1} \geqq \cdots \geqq \overline{n}_{gk_g} \geqq 0;\ n_{g1} - \overline{n}_{g1} \geqq \cdots \geqq n_{gk_g} - \overline{n}_{gk_g} \geqq 0.$$

If $\overline{n}_{gh} = 0$, the corresponding vectors $\overline{\varphi}_{ghij}$, understood to be 0, are to be omitted from the canonical basis $(\overline{\varphi}_{ghij})$; if every $\overline{n}_{gh} = 0$, the $\overline{m}$-space is simply the origin $(\overline{m} = 0)$. Further the canonical basis $(\overline{\varphi}_{ghij})$ may need for each g a slight notational readjustment in view of the possible equality of some of the numbers $\overline{n}_{g1}, \cdots, \overline{n}_{gk_g}$.

The invariant $\overline{m}$-space characterized by the integers $\overline{n}_{gh}$ consists precisely of the vectors φ of the m-space which are of the form

$$\varphi = \sum_{gh}\varphi_{gh} \quad (g = 1, \cdots, s;\ h = 1, \cdots, k_g),$$

where for every gh the vector φ_{gh} satisfies the conditions

(1) $$(T - w_gI)^{\overline{n}_{gh}}\varphi_{gh} = 0;$$

(2) there exists in the m-space a vector ψ_{gh} such that

$$(T - w_gI)^{n_{gh}-\overline{n}_{gh}}\psi_{gh} = \varphi_{gh}.$$

33. If a function $g(x, y, z)$ can be put into the form of a determinant of third order the elements of each of whose rows depend on one variable only, the values of the variables which satisfy the equation $g(x, y, z) = 0$ may be read off from a figure by drawing a straight line across three curves (d'Ocagne, Traité de Nomographie, page 132). As it is not always easy to decide whether $g(x, y, z)$ can be given the determinant form, it is desirable to have conditions which permit us to answer this question. Dr. Gronwall in his valuable paper in the *Liouville's Journal*, series 6, volume 8, page 59, has given such conditions in the form of the existence of a common solution of two partial differential equations of second order. The criteria obtained in Professor Kellogg's paper demand for their application only the operations of differentiation and of the determination of the ranks of matrices. Incidentally differential conditions for the linear dependence of a set of functions of several variables are given. The paper will be offered to the *Zeitschrift für Mathematik und Physik*.

34. At the beginning of Professor Glenn's paper he shows that, if a function f which satisfies certain postulates can be expressed as a finite series in one argument A, then there always exists a function Φ and an operator Δ capable of being represented symbolically in the form $\Xi \cdot \partial/\partial/H$ such that

$$f = \Phi + \Delta\Phi A + \frac{\Delta^2}{\underline{2}}\Phi A^2 + \cdots + \frac{\Delta^m}{\underline{m}}\Phi A^m.$$

This is also stated in homogeneous form, and generalized for p arguments. In the second section is treated the problem of expressing a form a_x^{mn} in p variables, as a finite expansion of order m in p arguments A_i $(i = 1, 2, \cdots, p)$, through the intermediary of the relations $A_i = \alpha_{ix}^n$, $(i = 1, 2, \cdots, p)$. The coefficients in this expansion are irrational fractional expressions in the resultants of the various forms involved, and the operators Δ of the general theory are the Aronhold operators obtained from α_{ix}^n. In the third section a polynomial $\alpha_x^m (m = \mu\nu)$ is expanded in a series of powers of a form ϕ_x^ν. Here Φ and also Δ are determined in general. The last section contains a new derivation of the Clebsch-Gordan expansion.

35. Struve's investigations have revealed two cases of

libration, one of them of unusual magnitude, among the inner satellites of Saturn; so far only the most important librational terms in the longitudes of these satellites have been investigated. It is the aim of the present paper by Professor Laves to make a consistent study of the motions of all the four satellites as a whole and to determine the secular and librational inequalities in the eccentricities, perisaturnia, inclinations, and nodes, and to trace their influence on the longitudes of the satellites.

36. In a paper just published in *Crelle's Journal* (volume 142, page 165) Fejér has given two different limiting processes by each of which the magnitude of a finite jump of a function may be obtained from its Fourier development. There is a third method of doing the same thing which is such an immediate and obvious consequence of the generalized statement of Gibbs's phenomenon given by Professor Bôcher in the *Annals of Mathematics*, volume 7 (1906), page 131, that it may be said to be substantially contained in that statement. If $f(x)$ has at x_0 a finite jump of magnitude D, one obtains in this way the formula

$$D = G \lim_{n=\infty} \left[S_n \left(x_0 + \frac{2\pi}{2n+1} \right) - S_n \left(x_0 - \frac{2\pi}{2n+1} \right) \right],$$

where S_n denotes the sum of the first $n+1$ terms of the Fourier development of $f(x)$, and where

$$\frac{1}{G} = 1 - \frac{2}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} \, dx.$$

This formula is similar to Fejér's formula (21), and perhaps even simpler than that formula. From it an analogue to his last formula on page 183 may at once be deduced. Very broad sufficient conditions on $f(x)$ for the correctness of these formulas are readily obtained, and generalizations are easily made.

37. In Science et Méthode, pages 158–160, Poincaré, in effect, constructs a general category of all inductive principles on the basis of resemblance. In this category occur the principle of "complete induction" and certain analogous

principles from which the former differs only by its certainty. The construction of the preceding general category depends itself upon the application of an inductive principle which seems not present in the category, The contradiction here presented by Mr. Schweitzer resembles the well-known contradiction of Richard.

38. The aims of Mr. Schweitzer's second paper are largely methodological: an attempt is made to describe the general logical position of mathematics as a heuristic science, to draw parallels between certain mathematical and philosophical authors, and to examine instances of the mathematical act. A definition of mathematics given by Peirce is adopted and is so interpreted as to recognize conflict as a part of mathematics. As leading examples of conceptual working hypotheses are discussed: (1) the principle of comparison, of which particular instances are the principles of Moore and Meinong; (2) the principle of continuation, of which a particular instance is the principle of permanence; (3) the principle of special situation; (4) the principle of the economy of thought and the working concepts beauty, elegance, simplicity, convenience, and naturalness. Leading examples of solutions of problems presented by conflicting terms in mathematics are mentioned, such as the "general analysis" of Moore, the "laws of thought" of Boole, the "extensive algebra" of Grassmann, etc. The main conclusions to which Mr. Schweitzer's investigation tends are stated thus: (1) The conceptual working hypotheses of mathematics are the same as those of non-mathematical disciplines; working hypotheses receive, however, a peculiarly mathematical character through discrimination exercised in their application. This discrimination is guided by perceptual working hypotheses: perceptual reconstructions of mathematical conceptions and primitive feelings or images. (2) There is essentially but one working hypothesis in mathematical procedure, viz., the principle of comparison.

39. If C_0 is the minimizing extremal for the calculus of variations problem

$$J = \int_{k_1}^{k_2} F(x, y, x', y')dt = \text{min.}, \quad K = \int_{k_1}^{k_2} G(x, y, x', y')dt = l$$

where k_1 and k_2 represent curves, and if C is any other curve connecting k_1 and k_2, satisfying the usual conditions, and for

which $K = l$, the total variation is $J_{\bar{c}} - J_{\bar{c}_0}$. It is the object of Dr. Crathorne's paper to express this total variation in a form somewhat analogous to the Weierstrassian E-function representation for the simple calculus of variations problem.

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CONCERNING TWO RECENT THEOREMS ON IMPLICIT FUNCTIONS.

BY DR. LLOYD L. DINES.

(Read before the American Mathematical Society, October 26, 1912.)

THE theorems here considered are two recent generalizations of the Weierstrassian implicit function theorem,* by Professor G. A. Bliss† and Mr. G. R. Clements.‡ They will be referred to respectively as Theorem B, and Theorem C.

The two theorems are similar in that they both give information concerning the number and character of the solutions of a system of equations

$$(1) \qquad f_i(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0 \quad (i = 1, 2, \cdots, p)$$

in the neighborhood of a point at which the functional determinant vanishes. They are different in that the assumptions concerning the functions f_i are different. As is so often the case with similarly related theorems, the ranges of applicability overlap, but neither is wholly contained in the other.§ The purpose of this note is to characterize explicitly the four classes of cases: (I) in which neither theorem is applicable; (II) in which both theorems are applicable; (III) and (IV) in which one theorem is applicable while the other is not.

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* Weierstrass, Abhandlungen aus der Funktionenlehre, p. 107.

† Bliss, "A generalization of Weierstrass' preparation theorem for a power series in several variables," *Transactions*, vol. 13, pp. 133–45 (April, 1912).

‡ Clements, "Implicit functions defined by equations with vanishing Jacobian" (Theorem IV), BULLETIN, vol. 18, p. 453 (June, 1912).

§ In presenting this note to the Society, I made the statement that Mr. Clements's theorem was a corollary of Professor Bliss's. That this statement was incorrect was pointed out to me by Mr. Clements, who exhibited a numerical example in which the hypothesis of his theorem was satisfied while that of Professor Bliss was not. The example comes under Case IV as treated in this paper.

In both theorems the functions f_i are supposed to be analytic in a neighborhood of the point in question, and that point is without loss of generality assumed to be the origin $(x) = 0$, $(y) = 0$. Each of the functions is therefore represented in a certain neighborhood of the origin by a convergent power series in $x_1, \cdots, x_n; y_1, \cdots, y_p$, with no constant term. If in any one of the series f_i the variables $x_1, \cdots, x_n$ are put equal to zero, the result is a power series in $y_1, \cdots, y_p$ with no constant term. For the discussion which follows we will adopt the notation

$$f_i(0, \cdots, 0; y_1, \cdots, y_p) \equiv f_i^{(k_i)}(y_1, \cdots, y_p) + f_i^{(k_i+1)}(y_1, \cdots, y_p) + \cdots,$$

where $f_i^{(g)}(y_1, \cdots, y_p)$ represents a homogeneous polynomial of degree g in $y_1, y_2, \cdots, y_p$. The leading polynomial $f_i^{(k_i)}(y_1, \cdots, y_p)$ has been called by Bliss (following MacMillan), the characteristic polynomial of the series f_i. The resultant of the system of p homogeneous polynomials $f_i^{(k_i)}$ will be denoted by $R[f_1^{(k_1)} f_2^{(k_2)} \cdots f_p^{(k_p)}]$.

With these conventions, the hypotheses of the two theorems in question can be stated as follows:

Hypothesis of Theorem B:

$$(B) \quad R[f_1^{(k_1)} f_2^{(k_2)} \cdots f_p^{(k_p)}] \neq 0, \quad \textit{and} \quad k = k_1, k_2, \cdots, k_p.$$

Hypothesis of Theorem C:

$$J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \qquad \textit{when } (x) = 0, \ (y) = 0,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$(C) \quad J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \qquad \textit{when } (x) = 0, \ (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \qquad \textit{when } (x) = 0, \ (y) = 0.$$

The *conclusions* of the two theorems may for our purpose be considered equivalent. Roughly speaking, both state that in the neighborhood of $(x) = 0$, equations (1) define $y_1, y_2, \cdots, y_p$ as *k-valued algebroid* functions of $x_1, x_2, \cdots, x_n$, vanishing when $(x) = 0$.*

* Theorem B states somewhat more than this, Theorem C somewhat less. However this conclusion can be obtained from conditions (C).

In order to make clear the relations between the hypotheses (*B*) and (*C*) and thereby the relations between the corresponding theorems, we fix attention on two simple conditions (*b*) and (*c*), which seem to contain the primary restrictions of conditions (*B*) and (*C*) respectively; they are:

$$(b) \qquad R[f_1^{(k_1)} f_2^{(k_2)} \cdots f_p^{(k_p)}] \neq 0,$$

$$(c) \qquad k_2 = k_3 = \cdots = k_p = 1.$$

In terms of these two conditions, the ranges of applicability of Theorems *B* and *C* are sharply defined by the following statements:

I. *In case neither* (*b*) *nor* (*c*) *is satisfied, neither Theorem B nor Theorem C is applicable.*

II. *In case both* (*b*) *and* (*c*) *are satisfied, both Theorem B and Theorem C* are applicable.*

III. *In case* (*b*) *is satisfied while* (*c*) *is not, Theorem B is applicable while Theorem C is not.*

IV. *In case* (*b*) *is not satisfied while* (*c*) *is satisfied, Theorem B is not applicable; but Theorem C may be, and is applicable if and only if there is an integer* $h(> k_1)$ *such that* $J_h(0, \cdots, 0; 0, \cdots, 0) \neq 0$.

The truths of these statements follow easily from two lemmas which we now prove.

Lemma I: *Conditions* (*C*) *can be satisfied only when* $k_2 = k_3 = \cdots = k_p = 1$.

By definition

$$J_k \equiv \begin{vmatrix} \dfrac{\partial J_{k-1}}{\partial y_1} & \dfrac{\partial J_{k-1}}{\partial y_2} & \cdots & \dfrac{\partial J_{k-1}}{\partial y_p} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} & \cdots & \dfrac{\partial f_2}{\partial y_p} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial f_p}{\partial y_1} & \dfrac{\partial f_p}{\partial y_2} & \cdots & \dfrac{\partial f_p}{\partial y_p} \end{vmatrix}.$$

Suppose now that one of the indices $k_2, k_3, \cdots, k_p$ is greater than 1. We may assume without loss of generality that it is k_2. Then the series $f_2(0, \cdots, 0; y_1, y_2, \cdots, y_p)$ begins with

* The fact included in II that when (*b*) and (*c*) are satisfied the hypotheses (*C*) are satisfied furnishes a generalization of Theorem VI of Mr. Clements's paper.

terms of higher degree than the first. Therefore all the elements in the second row of the determinant J_k must when $(x) = 0$ begin with terms of at least the first degree in y_1, y_2, $\cdots$, y_p, and hence when $(x) = 0$ and $(y) = 0$, the determinant J_k must vanish and (C) cannot be satisfied.

Lemma II: *If* $k_2 = k_3 = \cdots = k_p = 1$, *then* $J_g(0, \cdots, 0; 0, \cdots, 0) = 0$ *when* $g < k_1$, *and the condition* $J_{k_1}(0, \cdots, 0; 0, \cdots, 0) \neq 0$ *is equivalent to the condition* $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}] \neq 0$.

Since J_g is by definition a determinant in which the elements are power series, it is itself a power series, and we may use the notation

$$J_g(0, \cdots, 0; y_1, \cdots, y_p) \equiv J_g^{(j_g)} + J_g^{(j_g+1)} + \cdots,$$

where $J_g^{(h)}$ is a homogeneous polynomial of degree h. For the purpose of determining the value of $J_g(0, \cdots, 0; 0, \cdots, 0)$, it is sufficient to compute the leading polynomial $J_g^{(j_g)}(y_1, \cdots, y_p)$. The hypothesis $k_2 = k_3 \cdots = k_p = 1$ makes this computation simple.

From the definition of J_1 it follows that the first polynomial which can be different from zero in $J(0, \cdots, 0; y_1, \cdots, y_p)$ is

$$\begin{vmatrix} \dfrac{\partial f_1^{(k_1)}}{\partial y_1} & \dfrac{\partial f_1^{(k_1)}}{\partial y_2} & \cdots & \dfrac{\partial f_1^{(k_1)}}{\partial y_p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ a_{31} & a_{32} & \cdots & a_{3p} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{vmatrix}$$

where $a_{ij} = \partial f_i^{(1)} / \partial y_j$, that is a constant, since $f_i^{(1)}$ is a linear form. Since $f_1^{(k_1)}$ is of degree k_1, this determinant represents a polynomial of degree $k_1 - 1$. It can be expanded in the form

$$J_1^{(k_1-1)} \equiv \frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots + \frac{\partial f_1^{(k_1)}}{\partial y_p} A_p,$$

where $(-1)^{j-1} A_j$ is equal to the determinant obtained from the matrix

$$\left\| \begin{matrix} a_{21} & a_{22} & \cdots & a_{2p} \\ a_{31} & a_{32} & \cdots & a_{3p} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{matrix} \right\| \tag{2}$$

by striking out the jth column.

The first polynomial of $J_2(0, \cdots, 0; y_1, \cdots, y_p)$ which can be different from zero may be computed as follows:

$$J_2^{(k_1-2)} \equiv \begin{vmatrix} \frac{\partial J_1^{(k_1-1)}}{\partial y_1} & \frac{\partial J_1^{(k_1-1)}}{\partial y_2} & \cdots & \frac{\partial J_1^{(k_1-1)}}{\partial y_p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ a_{31} & a_{32} & \cdots & a_{3p} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{vmatrix}$$

$$\equiv \frac{\partial J_1^{(k_1-1)}}{\partial y_1} A_1 + \frac{\partial J_1^{(k_1-1)}}{\partial y_2} A_2 + \cdots + \frac{\partial J_1^{(k_1-1)}}{\partial y_p} A_p$$

$$\equiv \sum_{i=1}^{p} A_i \frac{\partial}{\partial y_i} \sum_{j=1}^{p} \frac{\partial f_1^{(k_1)}}{\partial y_j} A_j \equiv \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 f_1^{(k_1)}}{\partial y_i \partial y_j} A_i A_j$$

$$\equiv \left[\frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots + \frac{\partial f_1^{(k_1)}}{\partial y p} A_p \right]^{(2)}.$$

A simple mathematical induction gives for the first polynomial of $J_g(0, \cdots, 0; y_1, \cdots, y_p)$, $(g \leqq k_1)$ which can be different from zero, the formula

$$(3) \quad J_g^{(k_1-g)} \equiv \left[\frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots + \frac{\partial f_1^{(k_1)}}{\partial y_p} A_p \right]^{(g)}$$

If $g < k_1$, this polynomial is of first or higher degree and therefore $J_g(0, \cdots, 0; 0, \cdots, 0) = 0$; and the first part of the lemma is proved.

If $g = k_1$, formula (3) gives the constant term of J_{k_1}. Since $f_1^{(k_1)}$ is homogeneous of degree k_1, we have by Euler's theorem

$$\left[\frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots \frac{\partial f_1^{(k_1)}}{\partial y_p} A_p \right]^{(k_1)} = k_1! f_1^{(k_1)}(A_1, A_2, \cdots, A_p),$$

and therefore

$$(4) \quad J_{k_1}(0, \cdots, 0; 0, \cdots, 0) = k_1! f_1^{(k_1)}(A_1, A_2, \cdots, A_p).$$

In order to complete the proof of the lemma, it is sufficient, on account of (4), to show that $f_1^{(k_1)}(A_1, A_2, \cdots, A_p) = 0$ when and only when $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}] = 0$. Consider the

system of homogeneous equations

$$(5) \qquad \begin{aligned} f_1^{(k_1)}(y_1, y_2, \cdots, y_p) &= 0, \\ f_i^{(1)}(y_1, y_2, \cdots, y_p) &= 0 \end{aligned} \qquad (i = 2, 3, \cdots, p).$$

The last $p - 1$ of these equations are linear and the matrix of their coefficients is (2). Two cases are to be considered according as the rank of this matrix is less than or equal to $p - 1$.

If the matrix (2) is of rank less than $p - 1$, both $f_1^{(k_1)}(A_1, A_2, \cdots, A_p)$ and $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}]$ are equal to zero; the former because $A_1 = A_2 = \cdots = A_p = 0$, the latter because the polynomials $f_2^{(1)}, f_3^{(1)}, \cdots, f_p^{(1)}$ are not independent.

If the matrix (2) is of rank $p - 1$, then the solution of the last $p - 1$ equations of (5) is

$$y_j = \rho A_j \qquad (j = 1, 2, \cdots, p),$$

ρ being an arbitrary factor of proportionality. Therefore a necessary and sufficient condition that the system of p equations (5) have a solution in which not all of the variables are zero, is $f_1^{(k_1)}(A_1, A_2, \cdots, A_p) = 0$. But it is well known that a necessary and sufficient condition for the existence of such a solution is $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}] = 0$. Hence the two conditions are equivalent, and from (4) follows the desired conclusion of the lemma.

The statements I–IV will now be proved.

Proof of I: In case neither (*b*) nor (*c*) is satisfied: Theorem *B* is not applicable, since its hypothesis includes (*b*); and Theorem *C* cannot be applicable, by Lemma I.

Proof of II: In case both (*b*) and (*c*) are satisfied, both Theorems *B* and *C* are applicable. The hypothesis (*B*) is evidently satisfied, k being equal to k_1. And (*C*) is also satisfied, k being equal to k_1, as a direct consequence of Lemma II.

Proof of III: Since (*b*) is satisfied, (*B*) is satisfied, k being equal to $k_1 k_2 \cdots k_p$. Since (*c*) is not satisfied, (*C*) cannot be satisfied, by Lemma I.

Proof of IV: Since (*b*) is not satisfied (*B*) cannot be. Evidently (*C*) is satisfied if and only if there exists an integer h such that $J_h(0, \cdots, 0; 0, \cdots, 0) \neq 0$, k then being the smallest such integer. That k is necessarily greater than k_1 follows from Lemma II, and the fact that (*b*) is not satisfied.

CONCERNING THE PROPERTY Δ OF A CLASS OF FUNCTIONS.

BY PROFESSOR A. D. PITCHER.

(Read before the American Mathematical Society, April 26, 1913.)

IN the memoir* entitled "Introduction to a Form of General Analysis" E. H. Moore has studied real-valued functions and classes of real-valued functions of a general variable. He has given generalizations of numerous well-known theorems, has exhibited many new phenomena, and has indicated the important rôle this type of analysis is likely to play. The theory relates to properties of classes of functions. Some of these properties we define in the immediate sequel. In order to make application of the theory to a particular class of functions one must know whether or not the class possesses certain of the above-mentioned properties. One of the more difficult of these properties to study is the property Δ defined below (cf. the above memoir, § 79). The present paper establishes some theorems which are likely to be of service in this connection and which may be of interest in themselves. The page and section references of the sequel are all to Moore's memoir cited above.

A class $\mathfrak{M}$ of functions μ on a general range $\mathfrak{P}$ (page 4; § 4) is said to be *linear* (§ 14) in case every function of the form $a_1\mu_1 + a_2\mu_2$, where a_1, a_2 are arbitrary real numbers and μ_1, μ_2 are arbitrary functions of $\mathfrak{M}$, is of $\mathfrak{M}$.

A class $\mathfrak{M}$ is said to have the *dominance* property D if for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there is a sequence $\{a_n\}$ of real numbers and a function μ_0 of $\mathfrak{M}$ such that for every n and for every element p of the range $\mathfrak{P}$

$$|\mu_n(p)| \leqq a_n|\mu_0(p)|.$$

A *development*† Δ (§ 75) of a class $\mathfrak{P}$ of elements p is a system

$$((\mathfrak{P}^{ml})) \quad (m = 1, 2, 3, \cdots; \quad l = 1, 2, 3, \cdots, l_m)$$

* See New Haven Mathematical Colloquium, New Haven, Yale University Press, 1910.

† The following is a concrete example. Let $\mathfrak{P}$ be the class of all real numbers p such that $0 \leqq p \leqq 1$. Stage m of a development of $\mathfrak{P}$ is a set of $m + 1$ overlapping intervals of $\mathfrak{P}$. Thus for each m, $l_m = m + 1$, and for every m, l the class $\mathfrak{P}^{ml}$ is the interval $((l-2)/m, l/m)$ where $-1/m$ is taken as 0 and $(m+1)/m$ is taken as unity (§ 66a). A representative system for this development is the system $((r^{ml}))$ of numbers such that $r^{ml} = (l-1)/m$ (§ 66a).

of subclasses of $\mathfrak{P}$. For a given m the system $(\mathfrak{P}^{ml})$ is *stage* m of the development. A system $((r^{ml}))$ where r^{ml} is an element of $\mathfrak{P}^{ml}$ is said to be a *representative system* for the development Δ.

A class $\mathfrak{M}$ of functions on $\mathfrak{P}$ is said to have the *property* Δ (§§ 78, 79) relative to a development Δ of $\mathfrak{P}$ in case there is a system $((\delta^{ml}))$ of functions of $\mathfrak{M}$ such that there is a representative system $((r^{ml}))$ for the development Δ such that the following conditions are satisfied:*

(1a) For every positive number e there is a positive integer m_e such that for $m \geqq m_e$ and for every p

$$\left|\sum_g \delta^{mg}(p) - 1\right| \leqq e \quad \text{and} \quad \left|\sum_g |\delta^{mg}(p)| - 1\right| \leqq e.$$

(1b) For every μ there is a μ_0 such that for every e there is an $m_{\mu e}$ such that for $m \geqq m_{\mu e}$ and for every p

$$\sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq e|\mu_0(p)|.$$

The linear extension (§ 31) of a class $\mathfrak{M}$ of functions, denoted by $\mathfrak{M}_L$, is the class of all functions of the form $\sum_{i=1}^{n} a_i\mu_i$, where n is a positive integer, $a_1, a_2, \cdots, a_n$ are real numbers, and $\mu_1, \mu_2, \cdots, \mu_n$ are arbitrary functions of $\mathfrak{M}$. The $*$-extension (§ 27), denoted by $\mathfrak{M}_*$, of a class $\mathfrak{M}$ is a class composed of $\mathfrak{M}_L$ and the limit functions of all sequences of functions of $\mathfrak{M}_L$ which converge uniformly relative to a function of $\mathfrak{M}$ as scale function (§ 7d).

THEOREM I. *In case a class $\mathfrak{M}$ of bounded functions on a general range $\mathfrak{P}$ is linear and has the property D then if $\mathfrak{M}_*$ has the property Δ relative to a development of $\mathfrak{P}$ so does $\mathfrak{M}$ have the property Δ relative to the same development of $\mathfrak{P}$.*

* Relative to the development of the linear interval (01) given in the previous note and to the class of all continuous functions on this interval, a system $((\delta^{ml}))$ is given below (§ 66c). It will be seen that each δ^{ml} is a continuous function with a maximum value, unity, at the middle point of the interval $\mathfrak{P}^{ml}$ and with the value zero at every point outside of the interval $\mathfrak{P}^{ml}$. Thus

$$\delta^{ml}(p) = mp - l + 2, \qquad r^{ml-1} \leqq p \leqq r^{ml},$$
$$\delta^{ml}(p) = -mp + l, \qquad r^{ml} \leqq p \leqq r^{ml+1},$$
$$\delta^{ml}(p) = 0, \qquad p \leqq r^{ml+1};\ p \leqq r^{ml-1}.$$

Functions of this type have been quite generally used in analysis (cf. pp. 117, note).

To establish this theorem it is sufficient to exhibit a system $((\delta^{ml}))$ of functions belonging to $\mathfrak{M}$, and a representative system $((r^{ml}))$ such that conditions (1a) and (1b) are satisfied. Let $\mathfrak{M}_* \equiv \mathfrak{N} \equiv [\nu]$. Since $\mathfrak{N}$ has the property Δ there is a system $((\eta^{ml}))$ of functions belonging to $\mathfrak{N}$ and satisfying conditions (1a) and (1b) as stated for $\mathfrak{N}$. Thus

(2a) For every e there is an m_e such that for $m \geqq m_e$ and for every p

$$|\sum_g \eta^{mg}(p) - 1| \leqq e \quad \text{and} \quad |\sum_g |\eta^{mg}(p)| - 1| \leqq e.$$

(2b) For every ν there is a ν_0 such that for every e there is an $m_{\nu e}$ such that for $m \geqq m_{\nu e}$ and for every p

$$\sum_\eta |\nu(r^{mh})\eta^{mh}(p)| \leqq e|\nu_0(p)|.$$

Since $\mathfrak{M}$ is linear and has the property D there is for each ν a function $\bar{\mu}$ such that $|\nu| \leqq |\bar{\mu}|$ (§ 44a2) so that the following condition is satisfied.

(2′b) For every μ there is a $\bar{\mu}$ such that for every e there is an $m_{\mu e}$ such that for $m \geqq m_{\mu e}$ and for every p

$$\sum_h |\mu(r^{mh})\eta^{mh}(p)| \leqq e|\bar{\mu}(p)|.$$

Since the system $((\eta^{ml}))$ is of $\mathfrak{M}_*$ and $\mathfrak{M}$ is linear, there is for each η^{ml} a sequence $\{\mu_n{}^{ml}\}$ and a function μ^{ml} of $\mathfrak{M}$ such that as n increases without limit the sequence $\{\mu_n{}^{ml}\}$ approaches η^{ml} as a limit uniformly relative to the scale function μ^{ml} (§ 7d). Since $\mathfrak{M}$ has the property D, the system $((\mu^{ml}))$ of scale functions may be replaced by a single function $\hat{\mu}$ independent of m or l (§ 25.1). Since $\mathfrak{M}$ is composed altogether of bounded functions $\hat{\mu}$ may be taken so that $\hat{\mu} \leqq 1$. In each sequence $\{\mu_n{}^{ml}\}$ there is a function which may be denoted by δ^{ml} such that for every p

$$|\delta^{ml}(p) - \eta^{ml}(p)| \leqq \frac{1}{2^l \cdot m}|\hat{\mu}(p)|. \tag{3}$$

We proceed to show that, relative to the representative system $((r^{ml}))$ whose existence is postulated in (2b), the system $((\delta^{ml}))$ satisfies the conditions (1a) and (1b).

By means of (3) and the condition $|\hat{\mu}| \leqq 1$ it is easy to see

that for every p and m

$$(4)\qquad \begin{aligned} &\Big|\sum_g \delta^{mg}(p) - \sum_g \eta^{mg}(p)\Big| \leqq \frac{1}{m} \\ &\Big|\sum_g |\delta^{mg}(p)| - \sum_g |n^{mg}(p)|\Big| \leqq \frac{1}{m}. \end{aligned}$$

By means of (4) and (2a) it is readily seen that the system $((\delta^{ml}))$ satisfies condition (1a).

From (3) it follows at once that for every m, p, and h_p we have

$$(5)\qquad |\delta^{mh}(p)| \leqq \frac{1}{2^h m}|\hat{\mu}(p)| + |\eta^{mh}(p)|.$$

Thus for every μ and m and p we have

$$(6)\qquad \sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq \sum_h \frac{1}{2^h m}|\mu(r^{mh})\hat{\mu}(p)| + \sum_h |\mu(r^{mh})\eta^{mh}(p)|.$$

Since μ is bounded above by some positive constant a_μ we have

$$(7)\qquad \sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq \frac{a_\mu}{m}|\hat{\mu}(p)| + \sum_h |\mu(r^{mh})\eta^{mh}(p)|.$$

Since $\mathfrak{M}_*$ is linear and has the property D, the functions $\hat{\mu}$ of (7) and $\bar{\mu}$ of (2′b) may be replaced by a single function μ_0 (§ 24.1; § 22). Then from (7) and (2′b) it may be seen that the system $((\delta^{ml}))$ satisfies the condition (1b).

The above theorem may be useful in determining whether or not a given class $\mathfrak{M}$ of functions has the property Δ relative to a given development Δ of $\mathfrak{P}$ since the class $\mathfrak{M}_*$ is much more likely to contain a simple developmental system satisfying either or both of the conditions 1′a, 1′b of § 78.

It is true (§ 79.2) that if a class $\mathfrak{M}$ has the properties D, Δ then $\mathfrak{M}_*$ also has the property Δ. This combined with Theorem I gives the following theorem.

THEOREM II. *If a class $\mathfrak{M}$ of bounded functions μ is linear and has the property D then the necessary and sufficient condition that $\mathfrak{M}$ have the property Δ is that $\mathfrak{M}_*$ have the property Δ.*

If a class $\mathfrak{M}$ is composed of bounded functions and has the property D, the class M_L is linear and has the property D and is composed of bounded functions (§ 24.1; § 44a2, 5). Also

$(\mathfrak{M}_L)_* = \mathfrak{M}_*$ (cf. § 44*a*4). Also if $\mathfrak{M}_L$ has the properties D, Δ, so does $(M_L)_*$ (§ 79.2; § 44*a*5). In view of these propositions and Theorem II we have the following theorem.

THEOREM III. *If a class $\mathfrak{M}$ is composed of bounded functions μ and has the property D, then the necessary and sufficient condition that $\mathfrak{M}_L$ have the property Δ is that $\mathfrak{M}_*$ have the property Δ.*

In his dissertation, Chicago, 1912, E. W. Chittenden has made very effective use of infinite developments of a range $\mathfrak{P}$ where each stage of the development may contain a denumerably infinite number of subclasses. The theorems here given are valid also for such infinite developments. Theorem I may be established for infinite developments by essentially the same reasoning as above and in fact the same system $((\delta^{m\,l}))$ used above serves also in the case of infinite developments. The other theorems are established precisely as above.

DARTMOUTH COLLEGE,
February, 1913.

THE ASYMPTOTIC FORM OF THE FUNCTION $\Psi(x)$.

BY MR. K. P. WILLIAMS.

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THE function

$$\Psi(x) = -C - \sum_{s=0}^{\infty}\left(\frac{1}{x+s} - \frac{1}{s+1}\right),$$

where C is Euler's constant, is of great importance in many questions in analysis, and also in certain problems in mathematical physics. It is the logarithmic derivative of the gamma function, and plays a fundamental rôle in the study of the latter. On account of the slow convergence of the series which defines it, the knowledge of the asymptotic form of $\Psi(x)$ is particularly desirable.* This can be computed directly from the above expression by the aid of factorial series,†

* We use the term asymptotic according to the definition of Poincaré, and denote such a relation by the symbol $\sim$. See Borel, Les Séries divergentes, p. 26.

† Nielsen, Handbuch der Theorie der Gammafunktion, Kapitel XXI.

or it can be obtained from the definite integral form of the function.* Both of these methods, however, involve a considerable amount of complicated calculation.

We shall show in this paper how to obtain the asymptotic form of $\Psi(x)$ in a very simple manner from its fundamental functional property. As is well known, it is a solution of the non-homogeneous difference equation

$$f(x+1) - f(x) = 1/x, \tag{1}$$

and from this fact we shall derive the asymptotic form of the function, without making use of any, except one very obvious, property of its explicit analytical representation.

1. *The Formal Series.*—We shall merely assume that equation (1) has an analytic solution (all solutions differing from a particular one by a periodic function). It is immediately apparent from the equation itself that this solution increases over a set of points a unit's distance apart on the positive real axis in the manner of the harmonic series. From this we expect our solution to increase like $\log x$.

Let us therefore put

$$f(x) = \log x + a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots, \tag{2}$$

and substitute in (1). The quantity $\log(1+x) - \log x = \log(1+1/x)$ can be expanded in a series in $1/x$, which converges for $|x| > 1$. We have also for the same values of x

$$\frac{1}{(1+x)^n} = \frac{1}{x^n} \sum_{s=0}^{\infty} \frac{(-1)^s \binom{n-1+s}{n-1}}{x^s},$$

where, as usual,

$$\binom{m}{r} = \frac{m!}{r!(m-r)!}.$$

The expression which we obtain on substituting can thus easily be written as a series in $1/x$. When we equate the different coefficients to zero we find that a_0 is arbitrary, while

* Nielsen, loc. cit., Kapitel XIV.

$a_1, a_2, \cdots$ are given uniquely by the relations

$$a_1 + \frac{1}{2} = 0,$$

$$-\binom{2}{1}a_2 + a_1 + \frac{1}{3} = 0,$$

$$\binom{3}{2}a_3 - \binom{3}{1}a_2 + a_1 + \frac{1}{4} = 0,$$

$$-\binom{4}{3}a_4 + \binom{4}{2}a_3 - \binom{4}{1}a_2 + a_1 + \frac{1}{5} = 0,$$

. . . . . .

We shall next show how the a's as determined by the above equations are related in a simple manner to the Bernoulli numbers. Put

$$a_1 = -b_1,$$

$$a_n = \frac{(-1)^{n+1}}{n} b_n \qquad (n > 1).$$

The first of our series of equations then gives $b_1 = 1/2$, while from the $(n-1)$th equation we find

$$b_{n-1} + \frac{1}{2}\binom{n-1}{1}b_{n-2} + \frac{1}{3}\binom{n-1}{2}b_{n-3} + \cdots + \frac{1}{2}\binom{n-1}{1}b_2 - b_1 + \frac{1}{n} = 0.$$

When we add $2b_1 = 1$ to each member, and then multiply by n, this relation takes the form

$$\binom{n}{1}b_{n-1} + \binom{n}{2}b_{n-2} + \cdots + \binom{n}{n-2}b_2 + \binom{n}{n-1}b_1 + 1 = n.$$

Let us now introduce the symbol $\{X+1\}_n$ to represent the expression obtained on expanding $(X+1)^n$ by the binomial theorem, and then writing each of the exponents of X as a subscript. With this convention the above recursion formula between the b's takes the very simple form

$$\{b+1\}_n - b_n = n.$$

But this is precisely one of the symbolic equations which defines the Bernoulli numbers.* It therefore follows that all the b's of odd suffix, except b_1, and accordingly the corresponding a's, are zero.†

In order to make our results agree in notation with those usually given we shall put

$$b_{2n} = (-1)^{n+1}B_n,$$

so that $B_1, B_2, \cdots$ are the constants more commonly called the Bernoulli numbers.

We have thus completely determined all the a's except a_0, which was arbitrary, and to which we shall give the value zero. We then have as a formal solution of (1)

$$f(x) = \log x - \frac{1}{2x} + \sum_{s=0}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

where

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad \cdots.$$

2. *The Asymptotic Property of the Formal Series.*—We shall next investigate the relation between the formal solution we have obtained, and an actual solution of (1). In the first place, we see that the formal series diverges for all values of x; for when the subscript p is sufficiently large we have the inequality

$$\frac{B_{p+1}}{B_p} > A(2p+1)(2p+2),$$

where A is a constant.‡ It is therefore natural to study the difference between an actual solution and a certain number of terms of the formal solution.

Let us write

$$\varphi_n(x) = \log x - \frac{1}{2x} - \frac{B_1}{2x^2} + \cdots + (-1)^n \frac{B_n}{2n \cdot x^{2n}},$$

* Cesàro, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung. Deutsch von G. Kowalewski, p. 295.

† Cesàro, loc. cit., page 296.

‡ Borel, loc. cit., p. 24.

and then expand $\varphi_n(x+1) - \varphi_n(x)$ in a series in $1/x$. The series so obtained will converge for $|x| > 1$, and from the manner in which the a's in (2) were determined it is evident that the coefficients of $1/x^2, \cdots, 1/x^{2n+1}$ will all be zero,* the remaining coefficients, which we denote by $c_1^{(n)}, c_2^{(n)}, \cdots$, being uniquely determined in terms of $B_1, B_2, \cdots, B_n$. Thus we see that $\varphi_n(x)$ is a solution of the equation

$$(3) \qquad \varphi(x+1) - \varphi(x) = \frac{1}{x} + \frac{c_1^{(n)}}{x^{2n+2}} + \frac{c_2^{(n)}}{x^{2n+3}} + \cdots,$$

where the series converges outside a circle of unit radius.

Suppose next that $g(x)$ is any actual analytic solution of (1) and put

$$\theta_n(x) = \varphi_n(x) - g(x).$$

This new function is then a solution of the equation

$$(4) \qquad \theta(x+1) - \theta(x) = \frac{c_1^{(n)}}{x^{2n+2}} + \frac{c_2^{(n)}}{x^{2n+3}} + \cdots,$$

and consequently is given formally by

$$(5) \quad \theta_n(x) = \omega_n(x) - c_1^{(n)}\left[\frac{1}{x^{2n+2}} + \frac{1}{(x+1)^{2n+2}} + \cdots\right] - c_2^{(n)}\left[\frac{1}{x^{2n+3}} + \frac{1}{(x+1)^{2n+3}} + \cdots\right] \cdots,$$

where $\omega_n(x)$ is some periodic function of period 1.

To show that the double series above converges, we make use of the following inequalities:

$$(6) \qquad \sum_{s=0}^{\infty}\left|\frac{1}{(x+s)^k}\right| < \frac{\pi}{|x|^{k-1}} \qquad (k \geqq 2),$$

if x is in the right half of the complex plane and $|x| > 1$, and

$$(6') \qquad \sum_{s=0}^{\infty}\left|\frac{1}{(x+s)^k}\right| < \frac{2\pi}{|v|^{k-1}} \qquad (k \geqq 2),$$

* In the series from which we determined the a's by equating coefficients to zero, it is found that the coefficient of $1/x^{k+1}$ involves only $a_1, a_2, \cdots, a_k$.

if x is the left half plane* $(x = u + v\sqrt{-1})$ and $|v| > 1$. These inequalities, in connection with the fact that the series in (3) converges, show that the series in (5) represents an analytic function, and consequently the above expression defines $\theta_n(x)$, if $|x|$ is large enough and $\arg x \neq \pm\pi$. We see also that $\omega_n(x)$ must be analytic, since we are assuming the same to be true of $\theta_n(x)$.

We moreover have by (5) and (6) for x in the right half plane

$$|g(x) + \omega_n(x) - \varphi_n(x)| < \pi\left[\frac{|c_1^{(n)}|}{|x|^{2n+1}} + \frac{|c_2^{(n)}|}{|x|^{2n+2}} + \cdots\right].$$

Since the series is absolutely and uniformly convergent in any closed region exterior to the unit circle, this shows at once that

$$\lim_{x=\infty} x^{2n}(g(x) + \omega_n(x) - \varphi_n(x)) = 0,$$

when $-\pi/2 \leqq \arg x \leqq \pi/2$.

It will next be proved that the periodic function $\omega_n(x)$ is in reality independent of n. To show this we merely note that from the above

$$g(x) + \omega_n(x) - \varphi_n(x) \quad \text{and} \quad g(x) + \omega_{n+1}(x) - \varphi_{n+1}(x)$$

both approach zero for x large in the right half plane, while the quantity $\varphi_{n+1}(x) - \varphi_n(x)$ obviously itself approaches zero. It then follows immediately that the difference $\omega_{n+1}(x) - \omega_n(x)$ approaches zero when x goes to infinity along any ray making an acute angle with the positive real axis. But since this quantity is a periodic function, we must therefore have everywhere

$$\omega_{n+1}(x) = \omega_n(x).$$

We can now replace $\omega_n(x)$ by a unique periodic function $\omega(x)$. Then from the above limit and the definition of asymptotic representation we derive the important relation

$$g(x) + \omega(x) \sim \log x - \frac{1}{2x} + \sum_{s=1}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

for x in the right half of the complex plane.

* We can obtain these inequalities directly from those derived by elementary means by Birkhoff, "General theory of linear difference equations," *Trans. Amer. Math. Soc.*, vol. 12, p. 248.

If we should make use of (6′) instead of (6) we would find in a similar way that

$$\lim_{v=\infty} v^{2n}(g(x) + \omega(x) - \varphi_n(x)) = 0,$$

for x in the left half plane. But since $v = |x| \sin(\arg x)$, this limit reduces to the former, so that the above asymptotic relation holds for all approaches to infinity, provided only that $\arg x \neq \pm \pi$.

We can now state the theorem: *Given any analytic solution $g(x)$ of* (1), *there exists a unique analytic periodic function $\omega(x)$ such that $g(x) + \omega(x)$ is represented asymptotically by the formal solution of* (1), *for x approaching infinity in any direction except along the negative real axis.*

If instead of the value of $\theta_n(x)$ given by (5) we should take the solution of (4) which is analogous to the formal solution

$$l(x-1) + l(x-2) + \cdots$$

of the equation

$$h(x+1) - h(x) = l(x),$$

we could show in a similar way that there exists a periodic function such that its sum and the solution of (1) remains asymptotic to the formal solution for all approaches to infinity except the *positive* real axis.

Let us now take the solution $\Psi(x)$ of (1) and determine the corresponding periodic function. Let $x = n$, a positive integer; then

$$\Psi(n) = -C + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1},$$

whence we see that $\Psi(n) - \log n$ approaches zero when n increases indefinitely. It therefore follows that the periodic function must in this case be zero along the real axis and, being analytic, is consequently identically zero. We thus have the well known relation

$$\Psi(x) \sim \log x - \frac{1}{2x} + \sum_{s=1}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

for $-\pi < \arg x < \pi$.

It is evident that the above method and the relation

$\Gamma'(x)/\Gamma(x) = \Psi(x)$ give a very easy way to compute as many terms as may be desired in the series which occurs in the asymptotic form of the gamma function.

INDIANA UNIVERSITY,
March, 1913.

AN ERRONEOUS APPLICATION OF BAYES' THEOREM TO THE SET OF REAL NUMBERS.

BY DR. EDWARD L. DODD.

(Read before the American Mathematical Society, January 1, 1913.)

BAYES' theorem on the probability of causes is frequently introduced with an urn problem.* Here only a finite number of objects come into consideration. For example: The urn U_1 contained 3 white balls and 1 black ball; the urn U_2 contained 2 white balls and 2 black balls. A man, blindfolded, drew a white ball. What is the probability that this white ball came out of U_1,—assuming that each urn was equally accessible? After a consideration of the general problem of this nature, the following theorem, known as Bayes' theorem, is announced:

Let ω_i be the probability a priori that a certain urn, or "cause," or set of conditions U_i will come into play. The "causes" are to be mutually exclusive; and $i = 1, 2, \cdots, s$. Let p_i be the probability that U_i, when brought into play, will yield a certain event. Then, after this event has happened, the probability a posteriori P_i that the event had its origin in U_i is

$$P_i = \frac{\omega_i p_i}{\omega_1 p_1 + \omega_2 p_2 + \cdots + \omega_s p_s}.$$

In the preceding example, it is assumed that $\omega_1 = \omega_2 = \frac{1}{2}$. Hence, with $p_1 = \frac{3}{4}$, $p_2 = \frac{2}{4}$, it follows that $P_1 = \frac{3}{5}$,—a result which on inspection seems plausible; since $\frac{3}{5}$ of all the white balls were in the first urn, U_1. This urn example illustrates, indeed, the following important corollary of Bayes' theorem:

If each of a finite number s of mutually exclusive causes

* E. g., Poincaré, Calcul des Probabilités (1912), p. 153.

is equally likely a priori to come into play, then the probability a posteriori that a given event had its origin in a specified cause is proportional to the probability that the specified cause would produce that event.

In applying Bayes' theorem in the theory of errors of measurements, Poincaré* regards a possible value for the unknown as an ideal "cause." He lets z be any real number; and writes

$$\omega_i = \psi(z)dz$$

as the probability a priori that the true value of the unknown will lie between z and $z + dz$. With the true value in this interval, the probability that the first measurement will then lie between x_1 and $x_1 + dx_1$ is written as

$$dx_1\varphi(x_1, z).$$

If, now, the n measurements have been found to lie between x_1 and $x_1 + dx_1, \cdots, x_n$ and $x_n + dx_n$, respectively,—more briefly: to be $x_1, x_2, \cdots, x_n$, respectively,—then the probability a posteriori that the true value lies between z and $z + dz$ is found to be

$$\frac{dz\,\psi(z)\varphi(x_1, z)\varphi(x_2, z)\cdots\varphi(x_n, z)}{\int_{-\infty}^{+\infty} dz\,\psi(z)\varphi(x_1, z)\varphi(x_2, z)\cdots\varphi(x_n, z)}.$$

Poincaré then points out how Gauss by taking $\psi = 1$, by assuming that φ is a function of $z - x_i$, and by assuming that the arithmetic mean is the most probable value of the unknown, reaches the conclusion that

$$\varphi(y) = \sqrt{\frac{h}{\pi}}e^{-hy^2}, \tag{1}$$

where y is the error, $z - x$.

Poincaré then† mentions Bertrand's objections to the above assumptions, saying in part: "De plus, on a fait $\psi(z) = 1$, et l'on ne peut l'affirmer *a priori*."

We may add that the assumption that $\psi(z)$ is any constant for all real values of z is not only unwarranted, but it is analytically impossible. For if z is to have the range from $-\infty$

* Loc. cit., p. 169.
† P. 173.

to $+\infty$, the probability function $\psi(z)$ must satisfy the condition

$$\int_{-\infty}^{+\infty} \psi(z)dz = 1,$$

since it is certain that the number expressing the measure of the quantity lies between $-\infty$ and $+\infty$. *This condition cannot be satisfied if ψ is a constant.*

This equation, ψ = constant, seems to be the symbolic equivalent of the statement that, before measurements are made, all real numbers have equal probability a priori of being the true value,—if, indeed, we can attach any useful meaning to this statement.

The difficulty of so doing appears to have its origin in the fact that the set of real numbers is unbounded; rather than that the set contains an infinity of individuals. To illustrate: Suppose that a rod of unknown length lies in a narrow box with interior length of 20 inches. We may, then, with some propriety, set $\psi(z) = 1/20$; and let z range from 0 to 20. By so doing we would at least satisfy the requirement,

$$\int \psi(z)dz = 1,$$

taken over all the values of z possible in the given case. Furthermore, this probability may be interpreted as an ideal frequency. For instance, the probability that the length lies between 11.2 inches and 11.3 inches would be 1/200. This would signify that in 100,000 such boxes put up with rods taken by chance,* we should expect to find about 500 boxes containing rods with length between 11.2 inches and 11.3 inches.

The extension of these conceptions to the range from $-\infty$ to $+\infty$ does not seem to be immediate. Is the probability that the true value lies between 50 inches and 51 inches the same as the probability that the true value lies between one light-year and one light-year plus one inch? If so what is the probability in each case? That the generalization in question should present some difficulty is not surprising, in view of the fundamental difference between a proper integral and an improper integral.

This leaves one more hiatus in the argument that attempts

* It is not asserted here that under these circumstances we *must* take $\psi(z) = 1/20$.

to deduce the probability law (1) from the so-called principle of the arithmetic mean,—as the "most probable value." The argument referred to is that which first sets up

$$F(z) = \varphi(z - x_1)\varphi(z - x_2) \cdots \varphi(z - x_n)dy_1dy_2 \cdots dy_n$$

as the probability that, with z as the true value, the measurements $x_1, x_2, \cdots, x_n$, *will be* made; then attempts to regard this same expression as also proportional—or even equal—to the probability that z is the true value, the measurements $x_1, x_2, \cdots, x_n$, *having been* made; and then sets $dF/dz = 0$.

UNIVERSITY OF TEXAS,
December, 1912.

SHORTER NOTICES.

Die partiellen Differential-Gleichungen der mathematischen Physik. Nach Riemann's Vorlesungen in fünfter Auflage bearbeitet von HEINRICH WEBER. Zweiter Band. Braunschweig, Vieweg und Sohn, 1912. xiv+575 pp. Unbound 15 marks, bound 16.80 marks.

THE first volume of the fifth edition of this classic work was reviewed in this BULLETIN, volume 18, page 87, and the fourth edition in volume 8, page 81. Little need be added to these. The most noteworthy addition to the present volume is the entire section 18, devoted to relativity. This section contains thirty pages. The introduction points out the nature of time and that relativity is not really concerned with time but with the measure of time, or rather with the connection between time and space quantity. The succeeding sections are sufficiently described by their titles: time and space in the stationary and the moving world; normal form of the transformation of axes; constant velocity of light; significance of the Lorentz transformation; the fundamental electromagnetic equations for bodies at rest; the fundamental electromagnetic equations for moving bodies; invariancy of the equations; explicit form of the equations; transformation of the force and the displacement; the Michelson-Morley experiment; application of the relativity theory to the Michelson-

Morley experiment. One might suggest that, aside from its interest in physics, there is no more reason to include the theory of relativity here than any other theory of the possible groups of transformations that the differential equations of physics will admit. A larger consideration of such groups would indeed not be out of place. The theory of integral equations would seem to be of a much greater practical value to the working physicist in solving differential equations, yet the space devoted to that is very meager indeed. The promised developments are only a few pages in the section on vibrating membranes, in which the problem is reduced to one of integral equations. Substantially no idea is given of the methods of integral equations, the nearest approach being Green's functions for this case.

The article upon propagation of waves in a gas is preceded by a new section on thermodynamics. As it occupies only four pages, it gives but a few theorems. However, all the article has been re-written, in order to present the matter more clearly from the point of view of the previous edition. The twenty-second and the twenty-third divisions have been combined into a single one. A few minor changes occur.

JAMES BYRNIE SHAW.

Theorie der Elektrizität. Erster Band: *Einführung in die Maxwellsche Theorie der Elektrizität.* Von Dr. A. FÖPPL, vierte, umgearbeitete Auflage herausgegeben von Dr. M. ABRAHAM. Leipzig und Berlin, Teubner, 1912. xviii+410 pp. 11 Marks.

THE second edition of this volume was reviewed in this BULLETIN, volume 11, page 383. The third edition has not been accessible to the reviewer, consequently he cannot make a complete comparison between the two. In the present edition the vector applications to mechanics have been cut to a minimum; ponderomotive and fictive tensions have been treated more extensively; the theory of electric waves has been made to include the skin-effect, and some consideration of wireless telegraphy; and the electrodynamics of moving bodies is developed as far as can be done without bringing in the atomistic theories that belong to the second volume,—however, sufficient is given for the application to the induction phenomena of electrotechnics. The definition of the vector

displacement has been modified to read thus:

$$\mathfrak{D} = \epsilon\mathfrak{E} \text{ instead of } \mathfrak{D} = \frac{\epsilon}{4\pi}\mathfrak{E},$$

which introduces a change in the equations of the displacement current.

The influence of the Maxwell theory is evident from the popularity of this (and other similar treatments) and no doubt will continue to grow.

JAMES BYRNIE SHAW.

NOTES.

THE Colloquium Lectures delivered at the Princeton meeting of the American Mathematical Society, September 15–17, 1909, by Professor GILBERT A. BLISS on "Fundamental Existence Theorems," and Professor EDWARD KASNER on "Differential-Geometric Aspects of Dynamics," have been published by the Society in a volume of about 230 pages. The book is now on sale; price to members of the Society, $1.00, to non-members $1.50. Orders should be addressed to the American Mathematical Society, 501 West 116th Street, New York.

THE April number (volume 14, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "A study of the circle cross," by J. L. COOLIDGE; "Projective differential geometry of developable surfaces," by W. W. DENTON; "The solutions of non-homogeneous linear difference equations and their asymptotic form," by K. P. WILLIAMS; "An application of finite geometry to the characteristic theory of the odd and even theta functions," by A. B. COBLE; "Conformal transformations on the boundaries of their regions of definition," by W. F. OSGOOD and E. H. TAYLOR.

THE April number (volume 35, number 2) of the *American Journal of Mathematics* contains the following papers. "The reducibility of maps," by G. D. BIRKHOFF; "The highest common factor of a system of polynomials in one variable,"

by L. L. DINES; "Linear mixed equations and their analytic solutions," by R. D. CARMICHAEL; "On the theory of linear difference equations," by R. D. CARMICHAEL; "On the product of two quadro-quadric space transformations," by Miss H. P. HUDSON; "On some topological properties of plane curves and a theorem of Möbius," by S. LEFSCHETZ; "On a flat spread, sphere geometry in odd-dimensional space," by J. EIESLAND.

THE commission of the Wolfskehl foundation of the Göttingen academy provided a series of lectures at Göttingen during the week April 21–April 26 on the kinetic theory of matter. Lectures on different phases of the subject were given by M. PLANCK, P. DEBYE, W. NERNST, M. v. SMOLUCHOWSKI, A. SOMMERFELD, H. A. LORENTZ. A 16-page prospectus of the course is contained in the last number of the *Jahresbericht*.

ANALYTICAL catalogue cards for both the German and the French encyclopedias of mathematics can be obtained from the Library of Congress. The price of the set as issued to March 20, 1913, is: author set, German, $2.60, French, $1.60; dictionary set, German, $4.08, French, $1.60.

THE firm of Martin Schilling in Leipzig has recently prepared the following mathematical models.

1. Generation of a hyperboloid of revolution by means of the rotation of a straight line or a space curve, by Professor K. DOEHLEMANN.
2. Three plaster models of surfaces of constant width, by Professor MEISSNER.
3. Three card-board models of Bessel functions of complex argument, with a description by Professor A. SOMMERFELD, by Mr. Fr. BERGMANN.
4. Gyroscopic apparatus, by Professor L. PRANDTL.

THE Francoeur prize (1000 fr.) of the academy of sciences of Paris has been awarded to Dr. A. CLAUDE, of the bureau of longitudes of Paris, for his researches in mathematics and astronomy.

AT the session of the mathematico-physical society of Kasan held December 14, 1912, the Lobachevsky prize for

1912 was awarded to Professor F. SCHUR, of the University of Strassburg, for his book: Grundlagen der Geometrie. Professor J. L. COOLIDGE, of Harvard University, received honorable mention for his Elements of Non-Euclidean Geometry.

THE royal academy of sciences of Bologna has received a request from Dr. A. MERLANI to repeat its prize problem of 1912, for the best solution of which he will present 500 lire. The problem proposed is the following:

"To present, in a critical historical manner the organic development of the theory of elliptic functions, and the various points of view under which the theory has been presented from the end of the eighteenth century until the present time. Indicate the influence the various developments have had on other branches of mathematics."

No member of the academy may compete, otherwise there are no restrictions. Competing memoirs should be plainly written in Italian and be in the hands of the secretary, under the usual conditions, before December 31, 1914.

INDIANA UNIVERSITY. Summer Quarter (June 19–September 3, 1913).—By Professor S. C. DAVISSON: Advanced calculus, five hours; Theory of functions, five hours.—By Professor D. A. ROTHROCK: Ordinary differential equations, double course, first half.—By Professor R. D. CARMICHAEL: Projective geometry, five hours; Foundations of mathematics, five hours.

THE following courses in mathematics are announced for the academic year 1913–1914.

INDIANA UNIVERSITY.—By Professor S. C. DAVISSON: Theory of functions, two hours; Ordinary differential equations, three hours (*a*, *w*).—By Professor D. A. ROTHROCK: Differential geometry, three hours.—By Professor U. S. HANNA: Theory of groups of substitutions, two hours.—By Professor R. D. CARMICHAEL: Theory of ordinary differential equations, three hours; Bessel, Laplace, and Lamé functions, three hours; Difference equations, two hours.—By Mr. K. P. WILLIAMS: Fourier series and integrals, three hours (*s*).

All courses continue throughout the year, except those marked a = autumn, w = winter, s = spring.

YALE UNIVERSITY.—By Professor J. PIERPONT: Theory of functions of a complex variable, two hours; Modern analytic geometry, two hours; Theory of differential equations, two hours; Non-euclidean geometry, two hours.—By Professor P. F. SMITH: Differential geometry, two hours (second term); Continuous groups, two hours (second term).—By Professor E. W. BROWN: Advanced calculus and differential equations, three hours; Statics and dynamics, two hours; Advanced and theoretical dynamics, two hours, first half-year; Periodic orbits, two hours, second half-year.—By Professor W. R. LONGLEY: Integral equations with applications, two hours; Potential theory and harmonic analysis, two hours.—By Professor W. A. WILSON: Theory of functions of real variables, two hours.—By Dr. G. M. CONWELL: Theory of finite groups, two hours.—By Dr. D. D. LEIB: Advanced algebra, two hours.—By Dr. H. F. MACNEISH: Integration of differential equations; Synthetic projective geometry, two hours.—By Dr. E. J. MILES: Calculus of variations, two hours.—By Dr. J. I. TRACEY: Analytic geometry, two hours.

THE following courses in mathematics are announced for the present semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Integral calculus, four hours; with exercises, two hours; Applications of elliptic functions, four hours; The fundamental theorem of projective geometry, two hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of algebraic equations, II, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Special problems in the theory of functions, four hours; Theory of elliptic functions, four hours; Seminar, two hours.—By Professor G. HETTNER: Infinite series, products and continued fractions, two hours.—By Professor J. KNOBLAUCH: Theory of determinants, four hours; Theory of curved surfaces, II, four hours; Theory of space curves, II, one hour.—By Professor R. LEHMANN-FILHÉS: Differential calculus, four hours; with exercises, one hour.—By Professor I. SCHUR: Ordinary differential equations, four hours; Theory of functions, I, four hours.—Dr. K. KNOPP: Analytic geometry, four hours; Theory of aggregates, two hours; Theory of entire transcendental functions, two hours.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Projective geometry, two hours; with exercises, one hour; Analytic geometry, four hours; with exercises, one hour.—By Professor O. HÖLDER: Ordinary differential equations, four hours; with exercises, one hour.—By Professor G. HERGLOTZ: Algebra, four hours; Higher problems of elementary mathematics, two hours; Introduction to recent literature in the theory of functions, two hours.—By Professor P. KOEBE: Theory of functions, four hours; Fourier series and definite integrals, two hours; Recent literature in the theory of functions, two hours.—By Dr. R. KÖNIG: Algebraic analysis, two hours.

UNIVERSITY OF STRASSBURG.—By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Seminar, two hours.—By Professor F. SCHUR: General theory of curves and surfaces, four hours; Selected chapters of projective geometry, two hours; Seminar, two hours.—By Professor J. WELLSTEIN: Matrices and their application to vector analysis and to integral equations, four hours; Encyclopedia of elementary mathematics, two hours.—By Professor R. v. MISES: Descriptive geometry, four hours; with exercises, two hours; Calculus of probabilities, two hours; Fundamental concepts in the technics of flight, one hour; Seminar, two hours.—By Professor P. EPSTEIN: Calculus of variations, two hours.

DR. A. ROSENBLATT has been appointed docent in mathematics at the University of Cracow.

DR. U. CISOTTI, of the University of Padua, has been appointed professor of mathematical physics at the University of Pavia.

MR. A. S. EDDINGTON, chief assistant at the Royal Observatory, Greenwich, has been elected to the Plumian professorship of astronomy in the University of Cambridge.

DR. P. BOUTROUX, of the University of Poitiers, has been appointed professor of mathematics at Princeton University.

AT the meeting of the American philosophical society held April 19, Professor L. P. EISENHART, of Princeton University,

was elected to membership, and Professor Sir JOSEPH LARMOR, of the University of Cambridge, was elected foreign member.

AT the meeting of the National academy of sciences held in Washington, D. C., April 24, Professor L. E. DICKSON, of the University of Chicago, and Professor A. O. LEUSCHNER, of the University of California, were elected to membership.

DR. C. T. SULLIVAN has been promoted to an assistant professorship of mathematics at McGill University.

PROFESSOR P. H. SCHOUTE, of the University of Groningen and author of the Geometrie der mehrdimensionalen Räume, died April 18 at the age of 67 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Lezioni di geometria proiettiva, dettate nella università di Napoli. 3a edizione, migliorata e corretta. Ristampa, con appendice. Napoli, Pierro, 1912. 8vo. 15+506 pp. L. 12.00

ARCHIMEDIS Opera omnia cum commentariis Eutocii. Iterum edidit J. L. Heiberg. Volumen II. Leipzig, Teubner, 1913. 8vo. M. 8.00

AUERBACH (F.). See TASCHENBUCH.

BERNSTEIN (S. I.). See PICARD (E.).

BOLYAI (J.). La science absolue de l'espace. Traduit de l'allemand par J. Houel, avec une notice sur la vie et l'oeuvre de J. et W. Bolyai par F. Schmidt. Paris, Hermann, 1911. 8vo. Fr. 4.00

BUKREIEF (B. I.). Elements of algebraic analysis. Lectures in the advanced course for women at Kief. (Russian.) (*Kief University Bulletin.*) Kief, 1912. 8vo. 6+224 pp. R. 2.00

CALDARERA (F.). Trattato dei determinanti. Palermo, Virzi, 1913. 8vo. 255 pp. L. 7.00

CAPITO (C. A. A.). A text-book of mathematics and mechanics. London, Griffin, 1913. 12mo. 398 pp. $4.00

DEMARTRES (G.). Cours de géométrie infinitésimale. Avec une préface de G. Darboux. Paris, Gauthier-Villars, 1913. 8vo. 10+418 pp. Fr. 17.00

ENRIQUES (F.). Les concepts fondamentaux de la science. Traduit de l'italien par L. Rougier. Paris, Dunod et Pinat, 1912. 18mo. Fr. 3.50

FAZZANI (G.). See FREYCINET (C. de).

FREYCINET (C. de). Dell'esperienza in geometria. Traduzione di G. Fazzani. Palermo, 1912. L. 2.00

GABEREL (L.). Géométrie analytique en nombres conjugués. (Diss.) Neuchâtel, 1911. 8vo. 20 pp.

GRAVE (D. A.). Encyclopedia of mathematics. A survey of its present state. (Russian.) Kief, Ogloblin, 1912. 8vo. 10+601 pp. R. 3.50

GUIMARAES (R.). Les mathématiques en Portugal. Appendice II. Paris, Gauthier-Villars, 1911. 8vo. 107 pp. Fr. 4.00

HADAMARD (J.). Notice sur les travaux mathématiques de 1884 à 1912. 2 parties. Paris, Hermann, 1912. 4to. 150 pp. Fr. 6.00

HALSTED (G. B.). On the foundation and technic of arithmetic. Chicago, Open Court Publishing Co., 1913. 133 pp. $1.00

HANKEL (H.). Theory of complex number systems, especially of ordinary imaginary numbers and of Hamilton's quaternions, with their geometric representation. Russian translation by students of the mathematical club of Kazan University, edited with notes by I. I. Parfnitief. Kazan, 1912. 8vo. 16+242+3 pp.

HEIBERG (J. L.). See ARCHIMEDIS.

HERBST (W.). Mongesche Gleichungen zweiten Grades als Schnittbedingungen von Kurvenscharen. (Diss.) Greifswald, 1912. 8vo. 50 pp.

HEIBERG (J. L.). See ARCHIMEDIS.

HOUEL (J.). See BOLYAI (J.).

HRONYECZ (G.). Herleitung der Fuchsschen Periodenrelationen für lineare Differentialsysteme. (Diss.) Giessen, 1912.

IPATOF (V. M.). The elements of infinitesimal analysis, with a collection of exercises. For the seventh class of the "real" school. (Russian.) Moscow, Sytin, 1912. 8vo. 200 pp. R. 1.00

KLEIN (F.). Questions of elementary and higher mathematics. Lectures delivered at the University of Göttingen. Part I: Arithmetic, algebra, analysis. Russian translation by D. A. Kryzhanofsky, edited by V. F. Kahan. Odessa, *Mathesis*, 1912. 8vo. 19+486 pp. R. 3.00

KORKIN (A. N.). Works, edited by V. Steklof and A. Markof. (Russian.) Vol. I. (Published by the Physico-Mathematical Faculty of the University of St. Petersburg.) St. Petersburg, 1911. 8vo. 5+469 pp.

KREBS (H.). Théorie des groupes de transformations à un paramètre. (Diss.) Neuchâtel, 1911. 8vo. 16 pp.

KRYZHANOFSKY (D. A.). See KLEIN (F.).

LECAT (M.). Bibliographie du calcul des variations. Paris, Hermann, 1912. 8vo. 70 pp. Fr. 3.00

LETZ (E.). Die Verfolgungskurve des Kehlkreises auf den Rotationsflächen konstanter Krümmung. (Diss.) Halle, 1912. 8vo. 104 pp.

LIETZMANN (W.). The Pythagorean proposition, with a general view of Fermat's problem. Russian translation. (Library of elementary mathematics, edited by S. O. Shatunofsky. No. 1.) Odessa, *Mathesis*, 1912. 16mo. 2+80 pp. R. 0.40.

LILJESTRÖM (A.). Etudes sur la théorie du potentiel logarithmique. Stockholm, 1912. M. 1.50

LINDOW (M.). Differential- und Integralrechnung mit Berücksichtigung der praktischen Anwendungen in der Technik. (Aus Natur und Geisteswelt. Nr. 387.) Leipzig, Teubner, 1913. 16mo. 7+111 pp. M. 1.25

LOBATSHEFSKY (N. I.). New foundations of geometry, with a complete theory of parallels. With the addition of a biographical sketch of the author and notes. (Russian.) (Kharkof Mathematical Library, Nos. 2–3.) Kharkof, 1912. 8vo. 33+234 pp., 4 plates. R. 1.00

MARCO (F. De). La quadratura del circolo, e come è stato risolto il problema dalla scienza. Bologna, Galleri, 1913. 8vo. 16 pp.

MEMORIAL VOLUME presented to Professor G. K. Suslof (1881–1911). (Russian.) (*Kief University Bulletin.*) Kief, 1911. 8vo. 8+404 pp.

MERTÉ (W.). Ueber Kurven sphärischer Krümmung. (Diss. Jena.) Weida, Thomas und Hubert, 1912.

MEYER (F.). Diskussion eines Systems von Rotationsflächen 2ten Grades. (Diss.) Bern, 1911. 8vo. 70 pp.

MOROZOF (N.). The function; exposition of the differential and integral calculus with some of its applications to physical science and geometry. A guide to the independent study of higher mathematical analysis. (Russian.) St. Petersburg and Kief, Sotrudnik, 1912. 8vo. 12+404 pp. R. 3.50

NETTO (E.). Elements of the theory of determinants. Russian translation, edited, with notes, by S. O. Shatunofsky. Odessa, *Mathesis*, 1912. 8vo. 8+156 pp. R. 1.20

NUGEL (F.) Die Schraubenlinien. Eine monographische Darstellung. (Diss.) Halle, 1912. 8vo. 88 pp.

PARFNITIEF (I. I.). See HANKEL (H.).

PICARD (E.). On the development of certain fundamental theories of mathematical analysis in the course of the last century. Three lectures, delivered at Clark University, Worcester, Mass., U. S. A., 5, 6, and 7 July, 1899. Russian translation by S. O. Bernstein. (Kharkof Mathematical Library, edited by D. M. Sintsof. Series B, No. 1.) Kharkof, 1912. 12mo. 100 pp. R. 0.50

POPRUZHENKO (M.). Contribution to the methodology of infinitesimal analysis in secondary schools. (Russian.) St. Petersburg, Pedagogical Magazine, 1912. 8vo. 9+91 pp. R. 1.00

POSSÉ (K. A.). Course in differential and integral calculus. 3d edition, revised and enlarged by the author. (Russian.) St. Petersburg, Berezofsky, 1912. 8vo. 10+850 pp. R. 6.00

RABINOWITSCH (J.). Beiträge zur Auflösung der algebraischen Gleichungen 5ten Grades. (Diss.) Bern, 1911. 8vo. 32 pp.

RICHARD (J.). Sur la philosophie de la géométrie. Chateauroux, 1911. 8vo. 75 pp. Fr. 3.00

ROSHTSHIN (P.). Course in differential and integral calculus. (Russian.) Part I: Differential calculus. St. Petersburg, 1911. 8vo. 780+4 pp. Part II: Integral calculus. St. Petersburg, 1912. 8vo. 508+2 pp. R. 6.50

ROTHE (R.). See TASCHENBUCH.

ROUGIER (L.). See ENRIQUES (F.).

SCHOUTE (P. H.). Analytical treatment of the polytopes regularly derived from the regular polytopes. Section I: The simplex, with plate. Amsterdam, 1911. M. 4.00

SHATUNOFSKY (S. O.). See NETTO (E.).

TASCHENBUCH für Mathematiker und Physiker. Unter Mitwirkung zahlreicher Fachgenossen herausgegeben von F. Auerbach und R. Rothe. Mit einem Bildnis Friedrich Kohlrauschs. 3ter Jahrgang 1913. Leipzig, Teubner, 1913. 8vo. 10+437 pp. Cloth. M. 6.00

TIEDEMANN (K.). Zur Theorie der Elimination. (Diss.) Königsberg, 1912. 8vo. 63 pp.

TOMASSETTI (M.). See VOLTERRA (V.).

VELMIN (V. D.). On the theory of residuals of the eighth order in algebraic domains. (Russian.) (*Warsaw University Bulletin.*) Warsaw, 1912. 8vo. 21+229 pp.

VÉRÉBRUSSOV (A.). Mémoire sur les classes des nombres complexes idéaux conjugués, avec l'application à la démonstration du dernier théorème de Fermat. Paris, Gauthier-Villars, 1912. 8vo. 18 pp. Fr. 2.00

VIVANTI (G.). Esercizi di analisi infinitesimale. Puntata I. Completo. Pavia, Mattei, 1912. 8vo. 9 + 410 pp. L. 15.00

VOLTERRA (V.). Leçons sur les équations intégrales et les équations integro-différentielles. Leçons professées à la Faculté des sciences de Rome en 1910. Publiées par M. Tomassetti et F. S. Zarlatti. (Collection de monographies sur la théorie des fonctions.) Paris, Gauthier-Villars, 1913. 8vo. 6+164 pp. Fr. 5.50

WOODS (B. M.). A discussion by synthetic methods of two projective pencils of conics. (Publications in Mathematics.) Berkeley, Cal., University of California, 1913. 4to. Pp. 55–83. Paper. $0.50

ZARLATTI (F. S.). See VOLTERRA (V.).

II. ELEMENTARY MATHEMATICS.

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